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THE RATIONALE OF CIRCULATING NUMBERS,

WITH THE INVESTIGATIONS OF
All the RULES and PECULIAR PROCESSES uſed in that Part of
DECIMAL ARITHMETIC.

TO WHICH ARE ADDED, SEVERAL CURIOUS
MATHEMATICAL QUESTIONS;

WITH SOME USEFUL REMARKS ON
Affected EQUATIONS, and the Doctrine of FLUXIONS.

ADAPTED TO THE USE OF SCHOOLS.

By H. CLARKE.

Omnia quæcunque a primæva rerum natura conſtructa ſunt, numerorum videntur ratione formata. Hoc enim fuit principale in animo conditoris exemplar.
BOETHIUS.

LONDON:
Printed for the AUTHOR; and ſold by Mr. MURRAY, No. 32, in *Fleet-Street*,
M DCC LXXVII.

[illegible]

TO

Thomas Butterworth Bayley, Eſq;

OF HOPE,

FELLOW OF THE ROYAL SOCIETY.

SIR,

IT gives me the higheſt Satisfaction and Pleaſure, that you have condeſcended to receive this my firſt Eſſay under your Protection. And all who are honoured with your Friendſhip, and are acquainted with your ſuperior Knowledge in polite and uſeful Learning, in which you have juſtly included the Science of Numbers, will be ſenſible of my Happineſs in being thus permitted to addreſs you.

Were my Abilities, Sir, equal to my Wiſhes, I could with Pleaſure dilate on thoſe many excellent Qualifications, adorned with the utmoſt Good-nature and Humanity, which have rendered your Character ſo conſpicuous. But, as I well know I ſhould fail in the Attempt, the only Uſe I can make of this Opportunity, is, to teſtify my Regard to ſo generous a Patron, by publicly acknowledging the many Favours which I, however undeſerving, have received at your Hands; and which I ſhall always remember with the ſincereſt Gratitude. I am,

SIR,

Your moſt obliged

And obedient Servant,

HENRY CLARKE.

PREFACE.

WHEN we conſider the Excellency and ſuperior Uſefulneſs of Decimal Arithmetic above all other Kinds of Computation, we ſhall readily allow, that an Attempt to render the more intricate Parts thereof clear and intelligible, not only merits the peculiar Attention of thoſe concerned in the Inſtruction of Youth, but is particularly intereſting to all others who deſire Accuracy in their Calculations; and is therefore ſo far from being an unneceſſary Work, that it appears to be of the greateſt Utility. But as I am ſenſible how extremely difficult it is, even in the beſt Performances of this Nature, to eſcape the Malevolence of thoſe, who fancy it their Intereſt to keep others in a long Dependence on themſelves; I ſhall beg Leave to obſerve, that I ſhall be well pleaſed, notwithſtanding their Cenſure, if my Deſign meets with the Favour of the Candid and Ingenuous, who, I am perſuaded, upon a ſufficient Peruſal, will acknowledge, that the Method here purſued is not only new, but that it is attended with all the Perſpicuity the Subject can admit of.

The principal Deſign then of this Treatiſe is, to retrench all thoſe Superfluities (as I may call them) which

Cunn and others have loaded the Theory of Circulating Decimals with; and to ſhew, that the whole Buſineſs depends upon, or may be deduced from, this one ſingle Operation, That of finding *a finite Vulgar Fraction equivalent to an infinite Repeating Decimal.* This being once underſtood, the Rationale of all the Rules will be obvious; and the Pupil will then go on, not only with Pleaſure, but with Speed: For as the Man, who is engaged in a Race with every Obſtacle removed from his Courſe, has undoubtedly the Advantage of him who muſt turn and wind to get clear of the Impediments; ſo not only in this, but in every Art and Science, when Difficulties are removed, as well as a conciſe general Rule pointed out, the Mind's chief Labour is accompliſhed. So Horace ſays,

> *Quicquid præcipies, eſto brevis: ut citò dicta*
> *Percipiant animi dociles, teneantque fideles.*

Yet that I might not be thought to affect an unintelligible Conciſeneſs, I have conſidered every Rule diſtinctly, illuſtrated them with proper Examples, and given the Inveſtigation ſymbolically; by which the Scholar may at one View comprehend the whole Proceſs.

A Queſtion may poſſibly ariſe with ſome, what Advantage will reſult from a farther Inveſtigation of the Nature of Decimals? Have not the moſt eminent Mathematicians written profeſſedly on the Subject, and done all that was uſeful or curious therein? In anſwer to which, it will be neceſſary to take a retroſpective View of the principal Authors who have treated upon Decimals;

mals; from whence it will clearly appear, that there is ſtill Room for farther Improvement. The firſt Specimen of Decimal Arithmetic that we meet with, is in the Aſtronomical Tables of Arzachel, a Moor, who was very eminent in Spain about the Beginning of the eleventh Century. They are adapted to the Meridian of Toledo; and as they are calculated for the Arabian Year of the Hegira, were probably originally written in Arabic: The Perſians, Moors, Arabs, and Saracens, being about that Period very famous for their Knowledge in Aſtronomy. In theſe Tables, the Places of the Heavenly Bodies are denoted by a *centeſmal* Diviſion of the great Circles of the Sphere, to which the Arabian Algorithm of Numbers was better accommodated than the Greek or Roman literal Notation which had been hitherto made Uſe of for the Egyptian Sexageſms in the Aſtronomical Tables of Ptolemy, Albategnius, Abenazra, and other ancient Writers. Gerard Voſſius informs us alſo of a Treatiſe entitled *De Algorithmo*, written by Johannes de Sacro Boſco, about the Middle of the twelfth Century, who made Uſe of a *centeſmal* Notation for the Extractions of the Square and Cube Roots. About the Year 1460, John Muller, ſometimes named Regiomontanus, publiſhed his Book *De Triangulis*, in which he had conſtructed a Table of Sines to the Radius 10,000,000; an Account of which may be ſeen in the *Opus Palatinum de Triangulis*, by Otho and Rheticus. The next Improvement in this Part of Arithmetic, we find in a Treatiſe entitled *Arithmetica Memorativa*, compoſed in Latin Verſe, by William Buckley, about the Year 1530, wherein he has given a Rule for extracting the Square Root of a Fraction; the Operation being nearly the ſame with the preſent Mode of extracting the

Square Root of a Surd Number, excepting that it is limited to a certain Number of Cyphers : The Rule, as corrected by Dr. Wallis is,

> *Quadrato numero**, *ſenas præfigito Ciphras :*
> *Productii Quadri, Radix, per mille ſecetur.*
> *Integra dat Quotiens ; & pars ita recta manebit,*
> *Radici ut være nè pars milleſſima deſit.*

The Denominator being written under this Number, expreſſes the Square Root of the Fraction. Afterwards Peter Ramus, in his Arithmetic, written about the Year 1570, and publiſhed by Schoner, ſhews the Method of approximating to the Square and Cubic Roots of Surd Quantities, by adding Punctuations of Cyphers, exactly in the Manner we now practice. But the firſt Treatiſe written profeſſedly on this Subject, was publiſhed at Leyden, 1585, by Simon Stevens, entitled *DISME*, or Decimals ; which he tells us in his Geography, he believes to have been in Uſe among the Indians, and other Eaſtern Nations, long before the Sexageſſimal Notation was introduced by Ptolemy, in the Time of M. Aurelius. After this Time, Decimals began to be frequently uſed in Arithmetical Calculations, and were particularly much advanced by Briggs and Gellibrand, in their *Trigonometria Britannica* ; by Oughtred, in his *Clavis Mathematicæ denuò limata* ; alſo Wingate, Baker, Kerſey, and ſeveral other Authors of leſs Note, all contributed towards their Perfection, in their different Treatiſes of Arithmetic. Yet we do not find, that any Regard had been paid to the

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* Referring to the Product of the Numerator and Denominator, mentioned in a former Rule.

Nature of *Infinite Circulating Decimals* before Dr. Wallis's Time. He was, in all probability, the firſt who diſtinctly conſidered this curious Subject, as he himſelf informs us in his Treatiſe of Infinites. But he has neither given the Demonſtrations, nor ſhewn their Application. The latter of theſe Defects, Mr. Brown, in his Decimal Arithmetic, and afterwards Mr. Cunn in his Treatiſe of Fractions, attempted to ſupply, by giving Rules for their Operations. The former indeed has done this only in one ſingle Caſe; but the latter has extended it to all Caſes. But as theſe are alſo wanting in the main Point, namely, a Demonſtration, and are moreover deſignedly expreſſed in ſuch a Manner, as to ſet the Rationale of the Thing as far out of View as poſſible; it is neceſſary that either the Memory muſt be loaded with every Rule, or the Book be continually at Hand. Several other Authors have treated on Circulating Decimals. Martin, in his Decimal Arithmetic, has given ſome practical Rules, but hath not ſufficiently demonſtrated them. Emerſon, in his Cyclomatheſis, is excellent in the Theory, but has omitted the practical Part. Pardon, Vyſe, Thompſon, and ſome others, have alſo touched on this Subject; but as they all ſeem to have borrowed from Cunn, they are in the ſame Predicament. Malcolm and Donn are the only Authors I know of, who have treated the Doctrine of Circulates in an intelligible Manner. The principal Objection to the former, is, that he is too conciſe, referring conſtantly to the Rules for Vulgar Fractions, by which that Perſpicuity, which is the very Eſſence of a Demonſtration, is loſt; and the latter has omitted ſeveral Caſes in Multiplication and Diviſion, which frequently occur in Practice, and is alſo too curſory to afford the

Learner a proper Idea of the Subject. All our later Books of Arithmetic paſs by the Doctrine of Repetends unnoticed.

I muſt here beg my Readers not haſtily to impute Arrogance to me, as if I rejected all that has been done on this Subject, or ſuppoſed myſelf capable of what ſo many Men of great Parts and great Learning ſeem to have come ſhort of. For I acknowledge myſelf indebted to moſt of the Authors I have juſt mentioned, particularly to Malcolm and Emerſon, who have furniſhed me with ſeveral uſeful Hints. And yet I can juſtly make the ſame Obſervation with the former Author, namely, That "the Rules I have given are chiefly the Effect of Speculation" made ſome Years ago on this Subject, before I had ſeen his Syſtem of Arithmetic, or even any of the before-mentioned Authors.

It is not impoſſible but an Objection may be raiſed by others, who have never adverted to the Subject propoſed, as that this part of Arithmetic is of little or no no Uſe, ſince all Decimal Operations may be performed ſufficiently near the Truth without it. Theſe Perſons I ſhall refer to the Practical Queſtions at the latter end of this Treatiſe, and only here obſerve, that thoſe Inſtances, among innumerable others which might be produced, ſerve to ſhew, that the common Way is very defective, though carried out to ſeven or eight Decimal Places, and that the Method of Circulates is abſolutely neceſſary, where any Degree of Accuracy is required. The Want therefore of a Treatiſe ſomewhat of the following Kind, which ſhould at once ſhew the particular Properties of Repeating Decimals, and the Inveſtigations of the Rules for Operation from *one certain general principle*, naturally gave Occaſion to the following Sheets. For as I had frequently obſerved in the Courſe of Teaching, that Youth, by their not ſeeing the Reaſon of the Thing, could not long retain the Rules in their Memory, I endeavoured to demonſtrate them *vivâ voce*, and then obliged the Scholar to give the Inveſtigation himſelf *ex ſcripto*; by which I could eaſily diſcern whether he fully comprehended the Nature of them. But as the Inveſtigation and Rule given by the Pupil were generally too prolix and incorrect to be inſerted in their Cyphering-books, I drew up the following Sketch, as a general Form by which they might correct theirs, if neceſſary. By this Method of Procedure, I have the Pleaſure to find, that a Boy, who is tolerably acquainted with the common Rules in Arithmetic, will readily acquire as clear a Knowledge of the Rules for Circulating Decimals, as he has of any common Operation in Whole Numbers.

But as the Operations of Circulates (as well as all other Arithmetical Calculations) are moſt eaſily performed by Logarithms, I have ſhewn the Method of finding the Logarithm of any Repeating Decimal; whereby the whole Buſineſs is greatly facilitated, and the Difficulty and Intricacy of the Rules by Common Arithmetic avoided. And, for the Amuſement of ſuch Pupils as have touched on the firſt Principles of Algebra and Geometry, I have inſerted a few Queſtions, chiefly Originals, with their Solutions; and ſome are given without Solutions, which are intended for the Exerciſe of thoſe that are farther advanced. I have alſo added ſeveral Remarks on thoſe Parts of the Mathematics which ſeem to the young Reader to be rather obſcure, namely, On Cardan's and Colſon's Theorems for Cubic Equations, wherein a very clear and conciſe Rule is given for extracting the Cubic Root of an impoſſible Binomial; by which Cardan's Theorem is rendered generally uſeful, in finding the Roots of an Equation when they are *all* real, as well as when there is but *one* real and two imaginary—On the improbability of obtaining general Formulæ for the Surſolid and other higher Equations—On the Method of tabulating Literal Equations, illuſtrated by Examples; from whence the Reverſion of a Series, however affected with Radicals may be eaſily performed—On the direct and inverſe Method of Fluxions, wherein the Principles are fully explained, and by avoiding all Metaphyſical Conſiderations, rendered clear to the loweſt Capacity. The whole Buſineſs of finding Fluxions is reduced to one general Rule; and the particular Forms of fluxionary Expreſſions are ſo diſtinguiſhed, that the Learner may almoſt immediately determine in what Manner the Fluent may be obtained—On the Correction of a Fluent, and the Reaſon of it—On Trigonometrical Fluxions, with their great Importance in Aſtronomy—On the Phænomena of Saturn's Ring, being a new and curious Analytical Solution of the Problem reſpecting the Times of its appearance and diſappearance; whereby is alſo exhibited a new Species of Curves, &c. which is extracted from a Treatiſe juſt publiſhed, entitled, *Eſſai ſur les Phénomènes relatifs aux diſparitions périodiques de l'anneau de Saturne.* By M. Dionis du Séjour, Fellow of the Royal Societies of London and Paris. In the Regiſter of the Royal Academy of Sciences at Paris, for 1775, we have the following Encomium on

this Work, *Tels ſont les objets que* M. Du Séjour *a traités dans ſon Ouvrage ; et l'on voit qu'il n'a rien laiſſé à déſirer ſur la théorie des phaſes de l'anneau de Saturne. L'élégance, la fineſſe et la ſimplicité des méthodes dont il a fait uſage, rendent cet Ouvrage très-intéreſſant pour les Géomètres ; et la diſcuſſion des Phénomènes depuis* 1600 *juſqu'en* 1900, *le rend néceſſaire aux Aſtronomes qui voudront dans la ſuite obſerver avec préciſion ces apparences ; ainſi nous croyons qu'il mérite d'être imprimé avec l'Approbation et le Privilège de l'Académie.* Signed by M. M. d'Alembert, Le Chevalier Borda, Bézout, Vandermonde, M. de la Place, and Jean de Fouchi ; ſome of the greateſt Mathematicians now living *.—I have alſo added ſome new and uſeful Geometrical Propoſitions ; and, laſtly, have given a Catalogue of the moſt approved Authors in the ſeveral Branches of Mathematics, Philoſophy, and Aſtronomy ; from which are ſelected thoſe that are generally eſteemed the moſt uſeful, and ranged in the Order they may be read by the young Student to the moſt Advantage. The known and received Terms in the Circulates I have retained, except in one or two Inſtances, where others offered themſelves which ſeemed much more ſignificant; as for the words *Pure Compound* and *Mixed Compound*, which evidently carry in them an Air of Abſurdity, I have ſubſtituted *Pure Multiple* and *Mixed Multiple*. And for the greater Elegance and Perſpicuity in the Operations, I have made uſe of the common accentual Daſh over the Repetend; by which alſo that Illegibility of the Figures, often cauſed by the other Method, is avoided.

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* In this Treatiſe of Sejour's we meet with the following Paragraph ; which, as it may, perhaps, convince thoſe who ſtill doubt the Exiſtence of Saturn's Satellites, I have taken the Liberty to inſert.—*Le quatrième Satellite fut d'abord découvert par M. Huyghens, le* 25 *Mai* 1655. *Les quatre autres Satellites ont été découverts ſucceſſivement par Dominique Caſſini ; le cinquième ſur la fin d'Octobre* 1671, *le troiſième le* 23 *Décembre* 1672, *le premier et le ſecond en* 1684. *Les Anglais conteſtèrent long tems l'exiſtence de ces Satellites. Ce ne fut qu'en* 1718, *que M. Pound ayant élevé au-deſſus du clocher de ſa Paroiſſe, l'excellent objectif de cent vingt trois pieds de foyer, donné par M. Huygens à la Société royale, on fut aſſuré pour la première fois en Angleterre, que Saturne avoit rèellement cinq Satellites. Lors de leur découverte, M. Caſſini les avoit nommés* Aſtra Lodoicea, *par alluſion à Louis XIV, à l'exemple des Satellites de Jupiter, que Galilée avoit nommés* Aſtra Medicea *L'on fit frapper en France une Médaille pour conſacrer cette découverte ; la Médaille repréſentoit Saturne accompagné de ſes cinq Satellites, avec l'exergue ſuivante ;* Saturni Satellites primùm cogniti.

The Succeſs I have met with in my own School, by uſing the following Rules for Circulating Decimals, and their Inveſtigations, together with the Utility of the Mathematical Remarks, in rendering thoſe intricate Affairs extremely eaſy and intelligible to the Learner, and I may with ſome Truth add that trite Apology, the Solicitations of ſeveral of my Friends, have induced me to publiſh them. And as I can manifeſtly have no lucrative Views by ſo ſmall an Affair; ſo neither have I any great Anxiety about its Reception. Should its Uſefulneſs to either Maſter or Scholar be a Reaſon for its ſurviving the Attacks of a carping *Zoilus*, I have acknowledged already, it would be no unpleaſing Event. Should the contrary happen, I muſt (as Mr. Harris obſerves) acquieſce in its Fate; and let it peaceably paſs to thoſe deſtined Regions, whither our modern Productions are daily paſſing:

——in vicum vendentem thus & odores.

HOR. Lib. ii. epiſt. 1.

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THE

RATIONALE

OF

CIRCULATING NUMBERS.

SECT. I.

The Theory of Circulates.

1. A Circulate, or recurring decimal, is that wherein one or more figures are continually repeated: they are diſtinguiſhed into ſingle and multiple, and theſe again into pure and mixed.

2. A pure ſingle circulate is that which repeats a digit only; as ·666 &c. and is marked thus $\cdot\acute{6}$.

3. A pure multiple circulate is that in which ſeveral figures repeat; as ·642642 &c. marked $\cdot\acute{6}4\acute{2}$.

4. A mixed ſingle circulate is one which conſiſts of a terminate part, and a ſingle repeating figure; as 4·333, &c. or $4\cdot\acute{3}$.

5. A mixed multiple circulate is that which contains a terminate part and ſeveral repeating figures; as $86\cdot\acute{3}2\acute{5}$.

6. That part of the circulate which repeats is termed the repetend; as in $2\cdot\acute{6}$, and $34\cdot\acute{6}4\acute{3}$, $\acute{6}$ and $\acute{6}4\acute{3}$ are the reſpective repetends.

7. In any pure circulate, the whole repeating part, being continued *ad infinitum*, is equal to a vulgar fraction, of which the numerator is the repeating number, having the decimal point removed as many places to the right-hand as there are figures in the repetend, and the denominator an equal number of *nines*. For in the pure multiple circulate $\cdot 325325$, &c. *ſine fine*, if we put $r =$ the repeating figures $\cdot 325$, and $c =$ the whole circulating part, and from c take the $\frac{1}{1000}$ part of itſelf, we ſhall have $c - \frac{c}{1000} = \cdot 325 = r$; that is,

$$\begin{array}{ll} \text{From} & \cdot 325325, \text{ \&c.} = c \\ \text{Take} & \cdot 000325, \text{ \&c.} = \frac{c}{1000} \\ \hline \text{Rem.} & \cdot 325 = c - \frac{c}{1000} = r; \text{ but } c - \frac{c}{1000} \end{array}$$

$= c \times \overline{1 - \frac{1}{1000}} = c \times \frac{1000 - 1}{1000} = c \times \frac{999}{1000}$; hence $c = \frac{1000 \times \cdot 325}{999} = \frac{325}{999} = \cdot 325325$, &c. *ad infinitum*, where there are as many *cyphers* and *nines* as repeating figures; which proceſs evidently holds good for any pure circulating numbers.

Examples where the circulation begins in the integral part.

Ex. (1.) $\acute{3}\cdot\acute{7} = \frac{370}{99}$. (2.) $\acute{2}\cdot 0\acute{6} = \frac{2060}{999}$. (3.) $\acute{4}2\cdot 6\acute{3} = \frac{426300}{9999}$, &c.

Examples where the circulation begins with the prime decimal.

Ex. (1.) $\cdot\acute{3} = \frac{3}{9}$. (2.) $\cdot\acute{6}7\acute{4} = \frac{674}{999}$. (3.) $\cdot\acute{4}78\acute{6} = \frac{4786}{9999}$. (4.) $\cdot\acute{0}6\acute{4} = \frac{64}{999}$. (5.) $\cdot\acute{0}08\acute{3} = \frac{83}{9999}$, &c.

Examples where there are Cyphers, *which do not repeat, betwixt the decimal point and the firſt ſignificant figure.*

Ex. (1.) $\cdot0\acute{4} = \frac{\cdot4}{9}$. (2.) $.00\acute{6} = \frac{\cdot06}{9}$. (3.) $\cdot000\acute{2}\acute{6} = \frac{\cdot026}{99}$. (4.) $\cdot00\acute{7}0\acute{6} = \frac{7\cdot06}{999}$. (5.) $\cdot00\acute{7}14\acute{5} = \frac{71\cdot45}{9999}$, &c.

8. Hence if the repetend (punctuated according to its places) be divided by as many *nines* as there are repeating figures, the quotient will give the whole circulating part, and the fraction is a finite expreſſion for the ſeries infinitely continued.

9. Hence alſo if any number be multiplied by an *unit* with as many *Cyphers* annexed as it contains places, and then divided by as many *nines*, it becomes a circulate; which is ſingle or multiple according to the places in the given number, and pure or mixed as we take the whole or part of the given number for a repetend.

Pure Circulates.

Example 1.

6 × 10 = 60, and 9)60.(6.666, &c.

Example 2.

·6 × 10 = 6, and 9)6(·666, &c.

Example 3.

24 × 100 = 2400, and 99)2400(24·24,24, &c.

Example 4.

·234 × 1000 = 234, and 999)234,(·234,234, &c.

Mixed Circulates.

Example 5.

4·26 × 10 = 42·6, and 9)42·6(4·7333, &c.

Example 6.

327 × 100 = 32700, and 99)32700(330,30,30, &c.

Example 7.

·2071 × 1000 = 207·1, and 999)207·1(·2073,073, &c.

10. Since $\frac{6}{9}$ = ·666, &c. $\frac{234}{999}$ = ·234234, &c. it is evident, that ·666, &c. × 9 = 6; ·234234, &c. × 999 = 234; from whence it appears, that if any pure circulate be multiplied by as many *nines* (conſidered as decimals) as it contains places, the reſult will be the ſame number complete and terminate.

11. And ſince ·9 = 1 — ·1, ·99 = 1 — ·01, ·999 = 1 — ·001, &c. it follows, that if any circulate be divided by 10, 100, &c. according to the places of the repetend, and the quotient ſubtracted from the given circulate, there will remain

the ſame number, terminate and complete, that conſtituted the repetend.

Example 1.

·674674, &c. ÷ 1000 = ·000674674, &c.
Then from ·674674674, &c.
take ·000674674. &c.

there remains ·674 - - - - - the repetend complete.

Example 2.

3·737, &c. ÷ 100 = ·03737, &c. and 3·757, — ·03737, &c. = 3·7.

Example 3.

·00666, &c. ÷ 10 = ·00066, &c. and ·0066, &c. — ·00066, &c. = ·006.

12. A vulgar fraction, of which the denominator is any number of *nines* not leſs than the number of ſignificant figures in the numerator, is equal to a pure circulate, its repetend being the ſignificant part of the numerator. And for the punctuation obſerve, when the numerator and denominator are integral, if the places of the former exceed thoſe of the latter, the exceſs is the number of integral places in the circulate; but if the places of the latter exceed thoſe of the former, the exceſs ſhews the number of *cyphers* to be prefixed to the numerator for the repetend; and if the places of both are equal, the circulation begins with the prime decimal. If the numerator conſiſt of integers and decimals, or decimals only, there will be as many terminate *cyphers* as decimal places, prefixed to the numerator for the circulate.

Ex. (1.) $\frac{2060}{999} = \acute{2}\cdot0\acute{6}$. (2.) $\frac{426300}{9999} = \acute{4}2\cdot6\acute{3}$.

(3.) $\frac{674}{999} = \cdot\acute{6}7\acute{4}$. (4.) $\frac{83}{9999} = \cdot\acute{0}08\acute{3}$.

(5.) $\frac{.06}{9} = \cdot00\acute{6}$. (6.) $\frac{7\cdot06}{999} = \cdot00\acute{7}3\acute{6}$.

The truth of which appears from its being plainly the reverſe of Art. 7.

13. If the numerator of a Vulgar fraction conſiſts of complete repetends, and the denominator hath as many *nines* as the figures in the numerator, one period of the repeating figures is equivalent to the whole fraction.

Ex. (1.) $\frac{2626}{9999} = \cdot\acute{2}62\acute{6}$ (Art. 12.) which is evidently $= \cdot\overset{\prime\prime}{2}\overset{\prime\prime}{6}$.

(2.) $\frac{434\cdot3}{9999} = \cdot0\acute{4}34\acute{3} = \cdot0\overset{\prime\prime}{4}\overset{\prime\prime}{3}$.

14. If a vulgar fraction hath a repeating numerator, and the denominator as many *nines* as ſignificant figures in the repetend, that fraction is equal to the ſum of two or more other fractions, which may be turned into circulates by Art. 12.

Ex. (1.) $\frac{4646}{99} = \frac{4600}{99} + \frac{46}{99} = 46\cdot\overset{\prime\prime}{4}\overset{\prime\prime}{6} + \cdot\overset{\prime\prime}{4}\overset{\prime\prime}{6}$.

(2.) $\frac{320320}{99} = \frac{320000}{99} + \frac{320}{99} = 3232\cdot\overset{\prime\prime}{3}\overset{\prime\prime}{2} + \cdot\overset{\prime\prime}{3}\overset{\prime\prime}{2}$.

(3.) $\frac{454\cdot5}{99} = \frac{450}{99} + \frac{4\cdot5}{99} = \acute{4}\cdot\acute{5} + \cdot0\overset{\prime\prime}{4}\overset{\prime\prime}{5}$.

15. A vulgar fraction, of which the numerator is *unity*, and denominator any number of *nines*, is equal to an infinite ſeries of fractions, the numerator of the firſt term being an *unit*, and denominator as many *cyphers* as there are *nines* in the given fraction, with an *unit* annexed; and the ſucceeding terms in a geometrical progreſſion, of which the common ratio is the firſt term.

Thus $\frac{1}{9} = \cdot 11111$, &c. $= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$, &c. $= \frac{1}{10} + \left.\overline{\tfrac{1}{10}}\right|^{2} + \left.\overline{\tfrac{1}{10}}\right|^{3}$, &c.

$\frac{1}{99} = \cdot 010101$, &c. $= \frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000}$, &c. $= \frac{1}{100} + \left.\overline{\tfrac{1}{100}}\right|^{2} + \left.\overline{\tfrac{1}{100}}\right|^{3}$, &c.

16. From whence it follows, that any circulating decimal may be conſidered as conſiſting of a ſeries of fractions, of which the numerators are the repeating number, and the denominators an *unit*, with as many *cyphers* annexed as expreſs the local value of the repetend.

Example 1.

$\cdot\acute{6} = \cdot 666$, &c. $= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000}$, &c. $= \cdot 6 + \cdot 06 + \cdot 006$, &c.

Example 2.

$\cdot\acute{4}\acute{6} = \frac{46}{100} + \frac{46}{1000}$, &c. $= \cdot 46 + \cdot 0046 + \cdot 000046$, &c.

Example 3.

$${\cdot}0\acute{8}\acute{5} = \frac{85}{1000} + \frac{85}{1000000}, \text{\&c.} = {\cdot}085 + {\cdot}00085 + {\cdot}0000085, \text{\&c.}$$

From whence it is alſo evident that any circulate, as $2{\cdot}\acute{3}$ will be equal $2 + {\cdot}\acute{3} = 2 + {\cdot}3 + {\cdot}03 + {\cdot}003$, &c. or, $4{\cdot}6\acute{2}\acute{1} = 4{\cdot}6 + {\cdot}021 = 4{\cdot}6 + {\cdot}021 + {\cdot}00021$, &c.

17. The ſum of an infinite ſeries of *nines* is equal to *unity* in the next left-hand place; thus, ·999, &c. = 1, ·0999, &c. = ·1, ·00999, &c. = ·01, &c. For it is evident $\cdot 9 \left(= \frac{9}{10}\right)$ wants only $\frac{1}{10}$ of *unity*, ·99 wants $\frac{1}{100}$, and ·999, wants $\frac{1}{1000}$, and ſo on. So that if the ſeries were infinitely continued, the difference between that ſeries and *unity* would be equal to an *unit* divided by infinity; which, from the doctrine of infinites, is known to be equal *nothing*.

18. Hence it appears, that any pure circulate, being multiplied by as many *nines* as there are repeating figures, will always carry the repetend to the next left-hand place.

Example 1.

$${\cdot}666 \times 9 = 5{\cdot}999, \text{\&c.} = 6. = {\cdot}6 \times 9 + {\cdot}6.$$

Example 2.

$${\cdot}2424, \text{\&c.} \times 99 = 23{\cdot}999, \text{\&c.} = 24. = {\cdot}24 \times 99 + {\cdot}24.$$

19. The circulating figures may be ſuppoſed to begin at any place in the repetend. For $2\cdot342342$, &c. is evidently $=$

$$2\,\frac{342}{999} = 2.3\,\frac{423}{999} = 2\cdot34\,\frac{234}{999} = 2.342\,\frac{342}{999}, \text{ \&c.}$$

20. The repeating figures of any circulating number may be conſidered as conſiſting of twice, or thrice, that number of figures; or any multiple thereof.

Thus in $5\cdot2434343$, &c. the repetend 43 having 2 places, may be conſidered as having 4, 6, &c. places. For ſince 43 is ſuppoſed to be repeated *for ever*, it is obvious, that $5\cdot2\acute{4}\acute{3} =$

$5\cdot2\acute{4}434\acute{3} = 5\cdot2\acute{4}343434\acute{3}$, &c.

21. Any number multiplied by an *unit*, with any number of *cyphers* annexed, and once ſubtracted, is manifeſtly the ſame as being multiplied by as many *nines* as there are *cyphers*. For $10a - a = 9a$, $1000b - b = 999b$.

22. A mixed circulate, ſingle or multiple, is equal to a vulgar fraction, of which the numerator is the given number multiplied by 10, 100, 1000, &c. (according to the number of repeating figures) and leſſened by its terminate part, conſidered in it's local value, and denominator as many *nines* as repeating figures.

Example 1.

$$4\cdot\acute{3} = \frac{39}{9}.$$

For $4\cdot\acute{3} = 4\,\frac{3}{9}$, (Art. 7.) $= \frac{4\times9+3}{9}$; but $4 \times 9 + 3 =$

$4{\cdot}\acute{3} \times 9$ (Art. 18.) $= 4{\cdot}\acute{3} \times 10 - 4{\cdot}\acute{3}$ (Art. 21.) $= 43{\cdot}\acute{3} - 4{\cdot}\acute{3} = 43 - 4 = 39 \therefore \frac{39}{9} = 4{\cdot}\acute{3}$.

Example 2.

$$5{\cdot}2\acute{4}\acute{3} = \frac{519{\cdot}1}{99}.$$

For $5{\cdot}2\acute{4}\acute{3} = 5.2 + {\cdot}0\acute{4}\acute{3}$ (Art. 16.) $= 5{\cdot}2\ \frac{4{\cdot}3}{99} = \frac{5{\cdot}2 \times 99 + 4{\cdot}3}{99}$, and $5{\cdot}2 \times 99 + 4{\cdot}3 = 5{\cdot}2\acute{4}\acute{3} \times 99 = 5{\cdot}2\acute{4}\acute{3} \times 100 - 5{\cdot}243 = 524{\cdot}3 - 5{\cdot}2 = 519{\cdot}1, \therefore 5{\cdot}2\acute{4}\acute{3} = \frac{519{\cdot}1}{99}$.

Or thus, $5{\cdot}2\acute{4}\acute{3} \times 10 = 52{\cdot}\acute{4}\acute{3} = \frac{5191}{99}$, (Ex. 1.); and dividing by 10, we have $5\ 2\acute{4}\acute{3} = \frac{519{\cdot}1}{99}$.

Example 3.

$$2\acute{3}5{\cdot}\acute{7} = \frac{235500}{999}.$$

For $2\acute{3}5\ \acute{7} = 235{\cdot}\acute{7}3\acute{5}$ (Art. 19.) $= \frac{235{\cdot}735 \times 100 - 235}{999} = \frac{235{\cdot}7 \times 1000 - 200}{999} = \frac{235500}{999}$.

Or thus, $2\acute{3}5{\cdot}\acute{7} \div 100 = 2{\cdot}\acute{3}5\acute{7} = \frac{2355}{999}$ (Ex. 1.) $=$ (by multiplying by 100) $\frac{235500}{999}$; as before.

Or thus, $2\acute{3}5{\cdot}\acute{7} = 200 + \acute{3}5{\cdot}\acute{7} = 200\,\frac{35700}{999} = \frac{235500}{999}$; as before.

Example 4.

$$\cdot 2\acute{3}\acute{5} = \frac{23{\cdot}3}{99}.$$

For $\cdot 2\acute{3}\acute{5} = \cdot 2 + \cdot 0\acute{3}\acute{5} = \cdot 2\,\frac{3{\cdot}5}{99} = \frac{\cdot 2 \times 99 + 3{\cdot}5}{99} = \frac{\cdot 2\acute{3}\acute{5} \times 100 - \cdot 2\acute{3}\acute{5}}{99} = \frac{2\acute{3}{\cdot}\acute{5} - \cdot 2}{99} = \frac{23{\cdot}3}{99}.$

Or thus, $\cdot 2\acute{3}\acute{5} \times 10 = 2.\acute{3}\acute{5} = (\text{Ex. 1.})\,\frac{233}{99}$; which being divided by 10 gives $\cdot 2\acute{3}\acute{5} = \frac{23{\cdot}3}{99}$; as before.

Example 5.

$$\cdot 02\acute{7}\acute{4} = \frac{2{\cdot}72}{99}.$$

For $\cdot 02\acute{7}\acute{4} = \cdot 02 + \cdot 00\acute{7}\acute{4} = \cdot 02\,\frac{\cdot 74}{99} = \frac{\cdot 02\acute{7}\acute{4} \times 100 - \cdot 02}{99} = \frac{2{\cdot}72}{99}.$

Or, $\cdot 02\acute{7}\acute{4} \times 100 = 2{\cdot}\acute{7}\acute{4} = \frac{272}{99}$; this divided by 100 gives $\frac{2{\cdot}72}{99}$; as before.

23. If the Denominator of a vulgar fraction, (not repeating in the numerator) conſiſting of *nines*, have leſs places than the ſignificant figures of the numerator; that fraction is equal to a mixed circulate, of which the repetend has the ſame places

as there are *nines* in the denominator; and its circulate value may be thus found: Expunge the denominator, and remove the decimal point as many places to the left-hand as there are *nines* in the denominator; then daſh off the ſame number of ſignificant figures for a repetend; and the remaining figures to the left will be the terminate part. Add the terminate part to the numerator of the given fraction, obſerving to encreaſe the right-hand place by 1, if there be 1 carried in the addition, the ſum being daſhed for a repetend, and the decimal point removed as before, will give the correct circulate equal to the given fraction. The reaſon of which is evident from the laſt article.

Example 1.

Let $\frac{519\cdot1}{99}$ be propoſed.

Which being ordered according to the rule will ſtand thus;

$$\frac{519\cdot1}{99},\ 519\cdot1,\ 5\cdot191,\ 5\cdot1\acute{9}\acute{1},\ 5\cdot191,\ +\ 5\cdot2 = 524\cdot3\ \therefore\ 5\cdot2\acute{4}\acute{3} = \frac{519\cdot1}{99}.$$

Example 2.

$$\frac{235500}{999},\ 235500,\ \acute{2}35\cdot\acute{5},\ 235500+200 = 235700\ \therefore\ \acute{2}35\cdot\acute{7} = \frac{235500}{999}.$$

Example 3.

$$\frac{2\cdot72}{99},\ 2\cdot72,\ \cdot02\acute{7}\acute{2},\ 2\cdot72 + \cdot02 = 2\cdot74\ \therefore\ \cdot02\acute{7}\acute{4} = \frac{2\cdot72}{99}.$$

24. Any circulating number being multiplied by a given number, the product will be a circulating number, of which the repetend will have the ſame places as before. For every repetend being equally multiplied muſt produce equal products.

Example 1.

$\cdot\acute{3} \times 2 = \cdot 6$ hence

$$\begin{array}{rrr} & & 6\\ & 6 & \\ 6 & & \\ \hline \multicolumn{3}{l}{\cdot 666, \&c. = \cdot\acute{6}} \end{array}$$

But, if the product conſiſt of more places than the given number, the overplus belongs to the firſt place of the next repetend.

Example 2.

$\cdot\acute{6} \times 4 = 2{\cdot}4$, hence

$$\begin{array}{rrrr} & & 2 & 4\\ & 2 & 4 & \\ 2 & 4 & & 2\\ \hline \multicolumn{4}{l}{2{\cdot}666 \ \&c. = 2{\cdot}\acute{6}.} \end{array}$$

Example 3.

$\cdot\acute{2}3\acute{4} \times 13 = 3{\cdot}042$, therefore

$$\begin{array}{l} 3042\\ 3042\\ 3\,042 \qquad\;\; 3\\ \hline 3{\cdot}045045045 \ \&c. = 3{\cdot}\acute{0}4\acute{5}. \end{array}$$

25. Any vulgar fraction, of which the numerator and denominator are prime to each other (neither 2 nor 5) being reduced to a decimal, will be a circulate, and the places of the

repetend always leſs than the number of *units* in the denominator. For it is evident, the remainder is always leſs than the diviſor, one of which muſt therefore return a ſecond time, when the quotient conſiſts of as many places as there are *units* in the diviſor, if not before.

Example 1.

Let the vulgar fraction $\frac{3}{7}$ be propoſed.

```
7)3·0(428571,428571,4, &c.
  28
  --
   20
   14
   --
    60
    56
    --
     40
     35
     --
      50
      49
      --
       10
        7
       --
        3 &c.
```

Here all the remainders are 1, 2, 3, 4, 5, 6, one of which (3) returns in the 7th place; therefore the repetend conſiſts of 6 places.

Example 2.

Let $\frac{9}{11}$ be propoſed.

```
11)9·0(81,81, &c.
   88
   --
    20
    11
    --
     9 &c.
```

Here the remainders are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; and as one of them returns in the 3d place, the repetend has but 2 places.

26. Any vulgar fraction, of which the denominator is a prime, (except 2 and 5) being thrown into a decimal, will have as many circulating figures as the denominator would require *nines* for a dividend till it terminates. For if we divide 1000 &c. by any number till 1 remains, the quotient will evidently begin to repeat; but 999 &c. is only 1 leſs than 1000 &c. therefore *nothing* muſt remain when the figures have once circulated.

It is evident from the laſt article, that, whatever the dividend be, provided it be prime to the diviſor, the number of circulating figures will be the ſame.

Example 1.

Let $\frac{3}{11}$ be propoſed.

11)99(9
99
—
0
—

Here are 2 nines uſed in the dividend, therefore the repetend has 2 places.

11)3·0(·2727 &c. $= \cdot\acute{2}\acute{7}$.
22
—
Proof 80
77
—
3 &c.

Example 2.

Suppoſe $\frac{6}{7}$ be given.

7)999999(142857. Here are 6 nines required: hence the
······
repetend conſiſts of 6 places.

7)6·0

```
          7)6·0(·857142,857 &c. = ·8́57142́.
            56
            ——
             40
             35
             ——
Proof         50
              49
              ——
               10
                7
               ——
                30
                28
                ——
                 20
                 14
                 ——
                  6 &c.
                  —
```

Example 3.

Let $\frac{483}{37}$ be propoſed.

37)999(27. Here are 3 nines made uſe of; therefore, the repetend has 3 places.

Proof

$37)483{\cdot}0$ &c.$(13{\cdot}054{,}054$, &c. $= 13{\cdot}\acute{0}5\acute{4}$.

27. If two numbers be prime to each other, the leaſt common dividend, or number which exactly contains them, is expreſſed by their product; but if they be not primes, expreſs them fractionally, and reduce the fraction to its loweſt terms, the reciprocal product of the numerator and denominator of the two fractions is the leaſt common dividend.

Example

Example 1.

Let the numbers propoſed be 9 and 7.

9 and 7 are prime to each other; therefore, $9 \times 7 = 63$ is the leaſt number which contains 9 and 7 without a remainder.

Example 2.

Let 8 and 12 be propoſed.

Theſe numbers are not primes; therefore, $\frac{8}{12}=\frac{2}{3}$ (in it's loweſt terms) and 2×12, or $8 \times 3 = 24$, the leaſt common dividend.

28. If there be three or more numbers, find the leaſt number diviſible by any two of them, then the leaſt number diviſible by this and any other, and ſo on; the laſt number found will be the leaſt common dividend.

Example.

What is the leaſt common dividend of 2, 3, 4, and 8? 2 is prime to 3, $\therefore 2 \times 3 = 6$; 6 is not prime to 4, $\therefore \frac{6}{4}=\frac{3}{2}$, and $2 \times 6 = 12$; 12 is not prime to 8, $\therefore \frac{12}{8}=\frac{3}{2}$, and $2 \times 12 = 24$, the leaſt common dividend. The reaſon of this is evident from Euc. 37, 38. 7.

29. If the denominator of a fraction be compounded of two or more different primes (neither 2 nor 5) the leaſt common dividend of all the numbers, for each ſingle prime (found by the two laſt Art.) will expreſs the number of circulating figures.

Thus let the propoſed fraction be $\frac{3}{77}=\frac{3}{7\times 11}$; then, ſince $\frac{3}{7}$ $=\cdot\acute{4}2857\acute{1}$ repeats in 6 places, and $\frac{3}{11}=\cdot\acute{2}\acute{7}$ repeats in 2 places, and the leaſt common dividend of 6 and 2 is 6; therefore, we may ſuppoſe them both to circulate in 6 places, (Art. 20.) But 99 is diviſible by 11; therefore 99,99,99, muſt alſo be diviſible by 11: and ſince 999999 is alſo diviſible by 7, it muſt therefore be diviſible by $7\times 11=77$; conſequently, the repetend will conſiſt of 6 places.

Example.

```
77)99999(12987,
    77
    --
    229
    154
    ---
     759
     603
     ---
      669
      616
      ---
       539
       539
       ---
       ...
```

Where there are 6 nines uſed before it terminates.

Hence the vulgar fraction $\frac{3}{2849}$ will have a repetend of 6 places. For $\frac{3}{2849}=\frac{3}{11\times 7\times 37}$; the repetend by 11, 7, and 37, hath 2, 6, and 3 places reſpectively; and the leaſt number diviſible by theſe is 6; which is the number of places in the repetend.

Example.

```
2849)999999(351
      8547
      ----
      14529
      14245
      -----
        2849
        2849
        ----
        ....
```

Where it terminates with 6 nines.

30. And generally, if the denominator of a vulgar fraction $\frac{N}{D}$ be compounded of the prime numbers A B C &c. which circulate in *a b c* &c. places reſpectively; and P be the leaſt number which *a b c* &c. meaſure; then will $\frac{N}{A\times B\times C\,\&c.} = \frac{N}{D}$ thrown into a decimal, repeat in P places, which is obvious from the laſt Article.

31. If any circulating numbers, conſiſting of pure or mixed repetends, of equal places, be added together, the ſum will have a repetend of the ſame number of places. For every column of repetends muſt evidently give the ſame ſum.

Example 1.

```
      13·054,054, &c.

       2·342,342, &c.
      ---------------
Sum,  15·396,396, &c.
```

Example 2.

```
       2·7134,7134, &c.
        ·5428,5428, &c.
      71·4371,4371, &c.
      -----------------
Sum,  74·6934,6934, &c.
```

32. If any multiple repetends contain a certain number of places, and the places of the other repetends are found in a Geometrical progreſſion, of any ratio, the ſum will have a repetend conſiſting of as many places as the greateſt term of the progreſſion. For ſuppoſe the greateſt repetend to circulate in 8 places, the next leſs repetend can only circulate in 4, and the next in 2 places, to be conterminous (Art. 20.); and 2, 4, 8 are in Geometrical proportion.

Example 1.

8·72,72,72,72,72 &c.
74·12 74,12 74,12 &c.
2·19 36 42 57,19 &c.
85·04 83 28 04,04 &c.

Here 8, 4, 2 are the places of the repetends; each of which may therefore be ſuppoſed to conſiſt of 8 places.

Example 2.

·234234234234,234 &c.
·410725410725,410 &c.
·983210645271,983 &c.
1·628170290231,628 &c.

Here the repetends conſiſt of 3, 6 and 12 places; which being in Geometrical progreſſion, may be conſidered as conſiſting of 12 places each; therefore the ſum conſiſts of 12 places.

33. If the repetends to be added have unequal places, ſuppoſe them all to begin together, (Art. 19.) then will the leaſt common dividend of thoſe numbers give the places of repeating figures in the ſum. For it is evident that each period of repetends muſt then terminate together. (Art. 20.)

Example 1.

$$\begin{array}{r} 72{\cdot}\acute{3}\acute{2} \\ 52{\cdot}\acute{6}4\acute{3} \\ \acute{1}{\cdot}0\acute{4} \\ \acute{4}{\cdot}71\acute{4} \\ \hline \end{array}$$

Theſe repetends have 2, 3, 4 places, and the leaſt number diviſible by theſe is 12; therefore each repetend may be ſuppoſed to conſiſt of 12 places; which evidently cannot begin together till all the numbers become circulates.

Example 2.

78·141414 &c.
3·817817 &c.
21·737373 &c.

Here the repetends have 2 and 3 places, and the leaſt number diviſible by theſe is 6, which therefore denotes the places of the repetend in the ſum.

34. The difference of any two circulating numbers, of which the repetends can be made conterminous, will alſo be a circulating number of the ſame places as the greater repetend. This plainly follows from the laſt Article.

Example 1.

From 82·8546,8546, &c.
Take 8·72,72,72,72, &c.
Rem. 71·1274,1274, &c.

The greater repetend conſiſts of 4 places; ſo does the remainder.

Example 2.

From 7·382,382,382,382, &c.
Take 4·125 368,125 368, &c.
Rem. 3·257 014,257 014, &c.

Where the remainder has 6 places, the ſame as the greater repetend.

Examples for the Learner's Exerciſe.

Required the finite values of the following infinite repeating decimals.

Ex. (1.) $\cdot\acute{3}$. (2.) $\cdot\acute{4}\acute{5}$. (3.) $\cdot\acute{7}4\acute{1}$. (4.) $\cdot\acute{5}68\acute{6}$. (5.) $\cdot\acute{0}\acute{7}$.
(6.) $\cdot\acute{0}04\acute{6}$. (7.) $\cdot\acute{0}00\acute{3}$. (8.) $\acute{2}\cdot\acute{6}$. (9.) $\acute{4}\cdot0\acute{7}$. (10.) $\acute{7}62\cdot\acute{3}$.
(11.) $\acute{1}00\cdot0\acute{6}$. (12.) $\cdot0\acute{7}$. (13.) $\cdot00\acute{3}$. (14.) $\cdot00\acute{3}0\acute{7}$.
(15.) $\cdot00\acute{6}3\acute{2}$. (16.) $\cdot0\acute{7}000\acute{4}$. (17.) $2\cdot\acute{6}$. (18.) $2\cdot3\acute{4}\acute{7}$.
(19.) $7\acute{2}3\cdot\acute{6}$. (20.) $\cdot6\acute{1}\acute{3}$. (21.) $\cdot03\acute{6}\acute{4}$. (22.) $\cdot005\acute{6}3\acute{2}$.
(23.) $1212\cdot\acute{1}\acute{2}+\acute{1}\cdot\acute{2}$. (24.) $\acute{7}\cdot\acute{6}+\cdot0\acute{7}\acute{6}$.

Required the law of the infinite ſeries, or circulate decimal, equivalent to each of the following vulgar fractions.

$\frac{3}{9}$; $\frac{45}{99}$; $\frac{741}{999}$; $\frac{5686}{9999}$; $\frac{7}{99}$; $\frac{46}{9999}$; $\frac{3}{9999}$; $\frac{260}{99}$; $\frac{4070}{999}$;
$\frac{7623000}{9999}$; $\frac{10006000}{99999}$; $\frac{\cdot7}{9}$; $\frac{03}{9}$; $\frac{3\cdot07}{9\ 99}$; $\frac{6\cdot32}{9\ 99}$; $\frac{7000\cdot4}{9999\ 9}$; $\frac{24}{9}$;
$\frac{232\cdot4}{99}$; $\frac{722900}{999}$; $\frac{60\cdot7}{99}$; $\frac{3\cdot61}{99}$; $\frac{5\cdot627}{999}$; $\frac{120120}{99}$; $\frac{767\cdot6}{99}$;

SECTION II.

Addition of Circulates.

CASE I.

When the numbers to be added are pure, or mixed ſingle *circulates.*

RULE.

MAKE them all end together; (Art. 33.) then add as in common numbers, only increaſe the right-hand place of decimals by as many *units* as there are *nines* in that column, and the laſt figure will be the repetend. Or, continue the repetends two or three places forward, and add as before, obſerving to reject the ſuperfluous Figures to the right-hand of the repetend.

Example.

$24{\cdot}6\acute{8}$		$24{\cdot}688\acute{8}$	88
$7{\cdot}23\acute{4}$		$7{\cdot}234\acute{4}$	44
$1{\cdot}6$		$1{\cdot}600\acute{0}$	00
${\cdot}7\acute{3}$	Reduced	${\cdot}733\acute{3}$	33
$28{\cdot}5\acute{6}$		$28{\cdot}566\acute{6}$	66
$4{\cdot}\acute{7}40\acute{1}$		$4{\cdot}740\acute{1}$	11
$19{\cdot}\acute{6}$		$19{\cdot}666\acute{6}$	66
	Sum	$87{\cdot}230\acute{1}$.	

For $19\cdot\acute{6} = 19\cdot666\acute{6}$; $28\cdot5\acute{6} = 28\cdot566\acute{6}$, &c. (Art. 4.) and $\acute{6}+\acute{1}+\acute{6}+\acute{3}+\acute{0}+\acute{4}+\acute{8} = \frac{6}{9}+\frac{1}{9}+\frac{6}{9}+\frac{3}{9}+\frac{4}{9}+\frac{8}{9}$ (Art. 7.) $= \frac{28}{9} = 3\frac{1}{9} = 3+\acute{1}$.

Examples for the Learner's Exercise.

(1.)	(2.)	(3.)	(4.)
$21\cdot4\acute{3}$	$716\cdot\acute{4}$	$1468.$	$6\cdot\acute{6}$
$67\cdot\acute{8}$	$271\cdot6$	$21\cdot\acute{4}$	$\cdot\acute{9}$
$2\cdot1\acute{4}$	$\cdot\acute{8}$	$871\cdot2\acute{3}$	$\cdot\acute{7}$
$66\cdot$	$4\cdot6\acute{2}$	$\cdot00\acute{1}$	$\cdot\acute{2}$
$4\cdot\acute{1}$	$4\acute{6}.$	$41\cdot\acute{2}$	$\acute{4}\cdot$
$34\cdot8\acute{1}$	$7\cdot1\acute{2}$	$6\cdot1\acute{8}$	$.0\acute{6}$
$68\cdot\acute{7}$	$\cdot08\acute{2}$	$129\cdot61\acute{4}$	$\cdot00\acute{5}$

CASE II.

If the numbers be pure, or mixed multiple *circulates.*

RULE.

Make them all end together, (Art. 33.) then add as in common numbers, only to the right-hand figure add as many units as are carried from the column where the circulation begins, and mark off the repetend from the same place. Or, continue each repetend two or three places forward as before, rejecting the figures to the right-hand of the place where they become conterminous.

Example.

$4{\cdot}\acute{8}7\acute{5}$		$4{\cdot}8\acute{7}5875875875\acute{8}$	75
$16{\cdot}\acute{3}\acute{2}$		$16{\cdot}3\acute{2}32323232\acute{3}$	23
$428{\cdot}7\acute{6}$		$428{\cdot}7\acute{6}6666666666\acute{6}$	66
$31{\cdot}\acute{2}1\acute{6}$	Reduced	$38{\cdot}2\acute{1}621621621\acute{2}$	16
$14{\cdot}\acute{6}18\acute{4}$		$14{\cdot}6\acute{1}84618461\acute{6}$	18
${\cdot}0\acute{4}\acute{1}$		${\cdot}0\acute{4}141414141\acute{1}$	41
${\cdot}\acute{4}14\acute{2}$		${\cdot}4\acute{1}424142414\acute{4}$	14
	Sum	$503{\cdot}2\acute{5}61085937\acute{2}$,	

The places of repetends are 1, 2, 3, 4, and the leaſt number diviſible by theſe is 12. (Art. 28.) Each repetend may therefore be ſuppoſed to conſiſt of 12 Places (Art. 20.) and the circulation muſt evidently begin from the column where they all begin to repeat together,

The reaſon of adding the units to the right-hand figure of the ſum, is evident from continuing the repetends a few places forward.

Examples for the Learner's Practice.

(1.)	(2.)	(3.)	(4.)
$\acute{7}4\cdot1\acute{2}$	$4\cdot1\acute{2}\acute{7}$	$127\cdot1\acute{4}$	$\cdot00\acute{1}\acute{7}$
$\acute{4}\cdot\acute{5}$	$16\cdot\acute{2}$	$6\cdot$	$1\cdot7\acute{1}$
$\cdot\acute{6}8\acute{2}$	$\cdot\acute{4}1\acute{2}$	$4\cdot1\acute{2}\acute{7}$	$\cdot1\acute{4}$
$\cdot\acute{4}$	$4\cdot12$	$28\cdot21$	$6\cdot21\acute{4}$
$\acute{3}\;2\acute{1}$	$\cdot\acute{0}1\acute{6}$	$6\cdot0\acute{0}\acute{2}$	$\cdot1\acute{8}\acute{2}$
$\acute{1}8\cdot\acute{4}$	$17\cdot\acute{4}\acute{1}$	$5\cdot1\acute{6}$	$4\cdot1\acute{2}$

A General Rule.

Having made the repetends conterminous, divide their ſum by as many *nines* as they contain places, and carry the quotient to the next left-hand column to where the circulation begins; then add as in common numbers, and put down the remainder for the repetend, obſerving to prefix cyphers (if neceſſary) to make it equal in places to the other repetends.

Example.

$67\cdot\acute{5}\acute{1}$		$67\cdot5\acute{1}\acute{5}$
$4\cdot\acute{1}$		$4\cdot1\acute{1}\acute{1}$
$23\cdot4\acute{0}\acute{1}$	Reduced	$23\cdot4\acute{0}\acute{1}$
$4\cdot6\acute{2}$		$4\cdot6\acute{2}\acute{2}$
$17\cdot0\acute{2}$		$17\cdot0\acute{2}\acute{2}$
$3\cdot8\acute{3}$		$3\cdot8\acute{3}\acute{3}$
	Sum	$120\cdot5\acute{0}\acute{5}$

Here the places of the repetends are 1, 2, and the leaſt number diviſible by theſe is 2, hence each repetend will have two places to become conterminous; and the ſum of the repetends is plainly $\frac{15}{99}+\frac{11}{99}+\frac{1}{99}+\frac{22}{99}+\frac{22}{99}+\frac{33}{99}=\frac{104}{99}=1\frac{5}{99}$ $=1+\acute{0}\acute{5}$, (Art. 12.).

Examples for the Learner's Exerciſe.

(5.)	(6.)	(7.)	(8.)
$4\cdot6\acute{1}7\acute{8}$	$1\acute{2}\cdot0\acute{1}$	$\acute{4}6\cdot\acute{1}$	$44\cdot\acute{4}$
$17\cdot\acute{4}$	$4\cdot1\acute{5}\acute{1}$	$\acute{2}\cdot\acute{1}$	$\acute{2}\cdot\acute{8}$
$5\cdot16\acute{2}$	$18\cdot\acute{2}$	$27\cdot$	$4\cdot1\acute{2}\acute{7}$
$\cdot0\acute{1}7\acute{1}$	$\cdot00\acute{8}$	$\cdot\acute{0}\acute{8}$	$\cdot00\acute{9}\acute{8}$
$2\cdot4\acute{5}$	$1\cdot2\acute{1}$	$4\cdot1\acute{6}\acute{4}$	$18\cdot\acute{1}$
$16\cdot81\acute{7}$	$42\cdot6\acute{8}\acute{4}$	$21\cdot6\acute{7}$	$\acute{1}\cdot6\acute{8}$

(9.)	(10.)	(11.)	(12.)
$\cdot7\acute{8}\acute{4}$	$24\cdot2\acute{4}$	$\cdot\acute{0}\acute{1}$	$21\cdot$
$\cdot\acute{4}\acute{8}$	$\acute{1}\cdot\acute{8}$	$41\cdot$	$2\cdot\acute{4}$
$\cdot\acute{7}$	$17\cdot\acute{8}\acute{1}$	$\cdot\acute{6}8\acute{2}$	$18\cdot\acute{8}\acute{1}$
$\cdot6\acute{8}\acute{1}$	$5\cdot48\acute{1}$	$18\cdot\acute{4}$	$\cdot\acute{0}1\acute{4}$
$\cdot\acute{2}41\acute{6}$	$76\cdot\acute{2}8\acute{1}$	$\cdot6871$	$\cdot0\acute{0}\acute{4}$
$\cdot000\acute{7}$	$\acute{7}\cdot4\acute{1}$	$2\acute{1}\cdot\acute{4}8$	$28\cdot6\acute{5}$

SECTION III.

Subtraction of Circulates.

CASE I.

If the numbers conſiſt of pure, or mixed ſingle *circulates.*

RULE.

MAKE them conterminous, and ſubtract as in common numbers; the right-hand figure will be the repetend. But if the repetend of the ſubducend be greater than that of the minuend, inſtead of 10 add only 9 to the leſs repetend and ſubtract as before. Or, continue the repetends two or three places forward, and ſubtract as in common numbers, rejecting the figures to the right-hand of the repetend.

	Ex. 1.	Ex. 2.		
From	$24{\cdot}6\dot{7}$	$71{\cdot}4\dot{2}$	Reduced	$71{\cdot}4\dot{2}$ 2
Take	$18{\cdot}4\dot{3}$	$38{\cdot}\dot{7}$		$38{\cdot}7\dot{7}$ 7
Rem.	$6{\cdot}2\dot{4}$			$32{\cdot}6\dot{4}$

For (Ex. 2.) $71{\cdot}4\dot{2} - 38{\cdot}7\dot{7} = 71{\cdot}4\frac{2}{9} - 38{\cdot}7\frac{7}{9}. = 32{\cdot}6\frac{4}{9}$

$= 32{\cdot}6\dot{4}$. (Art. 7.).

Examples for the Learner's Practice.

	(1.)	(2.)	(3.)	(4.)
From	$68{\cdot}43\dot{2}$	$7{\cdot}4\dot{1}$	$54{\cdot}2\dot{7}$	$271{\cdot}\dot{4}$
Take	$42{\cdot}2\dot{7}$	$2{\cdot}\dot{8}$	$31{\cdot}6\dot{2}$	$39{\cdot}8\dot{7}$

CASE II.

When the numbers are pure, or mixed multiple *circulate.*

RULE.

Make them both conterminous and ſubtract as in common numbers; the figures in the remainder, which ſtand under the given repetends, will be the repetend of the difference. If the repetend of the ſubducend be greater than that of the minuend, leſſen the laſt figure in the remainder by one, and it will be the true repetend. Or, continue the repetends a few places forward, and ſubtract as before.

Example 1.

From	$27{\cdot}\dot{2}8\dot{3}$	Reduced	$27{\cdot}2\dot{8}3283\dot{2}$ 83
Take	$13{\cdot}1\dot{2}\dot{2}$		$13{\cdot}1\dot{2}3232\dot{3}$ 23
		Remains	$14{\cdot}1\dot{6}0050\dot{9}$

Example

Example 2.

From	$1\acute{2}7\cdot4\acute{6}$	Reduced	$1\acute{2}7\cdot462746274\acute{6}$	27
Take	$\acute{4}8\cdot\acute{6}$		$\acute{4}8\cdot648648648\acute{6}$	48
		Remains	$\acute{7}8\cdot81409762 5\acute{9}.$	

In Ex. 2. the places of the repetends are 3, 4; and the leaſt number diviſible by theſe is 12; which is therefore the number of repeating figures in the remainder.

The reaſon of taking *one* from the laſt figure of the remainder is evident from continuing the figures a few places forward.

Examples for the Learner's Practice.

	(3.)	(4.)	(5.)	(6.)
From	$24\cdot\acute{1}\acute{7}$	$3\acute{2}\cdot1\acute{8}$	$1\acute{7}8\cdot\acute{1}$	$271\cdot\acute{6}\acute{4}$
Take	$12\cdot\acute{4}$	$\acute{8}\cdot4\acute{1}$	$2\acute{4}\cdot\acute{6}$	$\acute{4}1\cdot\acute{6}$

	(7.)	(8.)	(9.)	(10.)
From	$1\cdot\acute{4}\acute{6}$	$17\cdot$	$\acute{8}\cdot4\acute{1}$	$\cdot\acute{2}8\acute{1}$
Take	$\cdot\acute{0}\acute{8}$	$\cdot\acute{4}1\acute{2}$	$2\cdot4\acute{1}\acute{8}$	$18.$

SECTION IV.

Multiplication of Circulates.

CASE I.

When one factor is a terminate, and the other a pure circulate.

RULE.

MULTIPLY the repetend and terminate together; then remove the decimal point as many places to the right-hand as there are repeating figures, and divide the product by the ſame number of *nines*, continuing the diviſion till it repeats or terminates, which will then be the true product. Or, inſtead of dividing by the *nines*, add the product to itſelf, removing it as many places forward as exceed the number of places in the repetend by one; and thus proceed till the reſult laſt added be carried beyond the firſt: laſtly, add theſe ſeveral products together, beginning under the right-hand place of the firſt, the ſum will be the true product, having a repetend equal in places to the given circulate.

1. A ſingle terminate, and a pure ſingle circulate.

Example.

Multiply $\cdot\acute{8}$ by 7.

$$\begin{array}{l} \cdot 8 \\ \underline{7} \\ 9)\underset{\cdot\,\cdot}{56}(6\cdot22, \text{\&c.} = 6\cdot\acute{2} \end{array}$$

Or thus,

$$\begin{array}{r} \cdot 8 \\ 7 \\ \hline 56 \\ 56 \\ \underline{5} \\ 6\cdot\acute{2} \end{array}$$

For $\cdot\acute{8} \times 7 = \frac{8}{9} \times 7$ (Art. 7.) $= \frac{56}{9} = 56 \times \frac{1}{9} = 56 \times : \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$, &c. $= 5\cdot6 + \cdot56 + \cdot056$, &c.

Examples for the Learner's Practice.

Multiply $\cdot\acute{6}$ by 2. $\cdot\acute{5}$ by 4; $\cdot\acute{7}$ by 3; $\cdot\acute{8}$ by 5; $\cdot0\acute{7}$ by $\cdot04$; $\cdot00\acute{3}$ by 60; $\cdot00\acute{5}$ by 400; $\cdot000\acute{6}$ by 20.

2. A terminate conſiſting of ſeveral figures, and a pure ſingle circulate.

Example.

Multiply 23.4 by $\cdot\acute{7}$.

$$\begin{array}{rcr}
23\cdot4 & \text{Or thus,} & 16\cdot3\ \ 8 \\
\cdot7 & & 1\ 6\ \ 38 \\
\hline
 & & 1\ \ 63 \\
99)163\cdot8(18\cdot2 & & 16 \\
 & & 1 \\
\hline
 & & 18\cdot1\ \ \acute{9} = 18\cdot2, \text{ (Art. 17.)}
\end{array}$$

For $\cdot\acute{7} = \frac{7}{9} = 7 \times \frac{1}{9} = 7 \times : \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$, &c. (Art. 15.) $\therefore\ \cdot\acute{7} \times 23\cdot4 = 163\cdot8 \times : \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$, &c. $= 16\cdot38 + 1\cdot63 + \cdot1638$, &c.

For the Learner's Practice.

Multiply 21 by $\cdot\acute{3}$; 643 by $\cdot\acute{7}$; $\cdot265$ by $\cdot0\acute{6}$; $\cdot0281$ by $\acute{2}\cdot\acute{2}$; $4\cdot001$ by $\cdot00\acute{4}$.

3. A ſingle terminate, and a pure multiple circulate.

Example.

Multiply $\acute{3}\cdot4\acute{1}$ by $\cdot4$

$$\begin{array}{r} 3\cdot41 \\ \cdot4 \\ \hline \end{array}$$

999)1364(1·365,365, &c.

Or thus,

$$\begin{array}{r} 1\cdot364 \\ 13 \\ 1 \\ \hline 1\cdot\acute{3}6\acute{5} \end{array}$$

For $\acute{3}\cdot4\acute{1} \times \cdot4 = \frac{3410}{999} \times \cdot4 = \frac{1364}{999} = 1364 \times \frac{1}{999} = 1364 \times : \frac{1}{1000} + \frac{1}{1000000}, \&c. = 1\cdot364 + \cdot001364, \&c.$

For the Learner's Exerciſe.

Multiply $\acute{2}\cdot\acute{1}$ by 8; $\cdot\acute{7}1\acute{4}$ by ·6; $\acute{7}2\cdot\acute{1}$ by ·02; $\acute{4}6\cdot\acute{8}$ by ·004. $\cdot0\acute{0}4\acute{8}$ by ·4; $\cdot\acute{1}\acute{8}$ by ·01.

4. A terminate of ſeveral figures, and a pure multiple circulate.

Example.

Multiply 48·76 by $\acute{3}2\cdot4\acute{1}$.

$$\begin{array}{r} 48\cdot76 \\ 32\cdot41 \\ \hline \end{array}$$

9999)15803116(1580·4696,46, &c.

Or thus,

$$\begin{array}{r} 1580\cdot3116 \\ 15803 \\ 1 \\ \hline 1580\cdot\acute{4}69\acute{6} \end{array}$$

For $\acute{3}2{\cdot}4\acute{1} \times 48{\cdot}76 = \frac{324100}{9999} \times 48{\cdot}76 = \frac{1580{\cdot}3116}{9999} =$ $1580{\cdot}3116 + {\cdot}15803116 +$ &c.

For the Learner's Exerciſe.

Multiply 276. by $\acute{2}{\cdot}\acute{1}$; 4·68 by ${\cdot}\acute{0}\acute{4}$; ·071 by $\acute{4}{\cdot}8\acute{4}$; $\acute{2}8{\cdot}\acute{4}$ by 21; ${\cdot}\acute{0}0\acute{4}$ by 108; $\acute{4}.\acute{8}$ by ·006.

CASE II.

If one factor be a terminate, and the other a mixed circulate.

RULE.

Multiply the circulate by an *unit*, with as many *cyphers* annexed as the repetend has places, from which ſubtract the terminate part; then multiply the remainder by the given terminate, and divide the product by as many *nines* as the circulate has repeating figures; or, add the products as directed in Caſe I.

1. A ſingle terminate and a mixed ſingle circulate.

Example.

Multiply $4{\cdot}2\acute{3}$ by 6

$$
\begin{array}{r}
4{\cdot}23 \\
10 \\
\hline
42{\cdot}3 \\
4{\cdot}2 \\
\hline
38{\cdot}1 \\
6 \\
\hline
9)228{\cdot}6 \\
\hline
25{\cdot}4
\end{array}
\qquad \text{Or, thus,} \qquad
\begin{array}{r}
4{\cdot}23 \\
6 \\
\hline
25{\cdot}38 \\
2 \\
\hline
25{\cdot}40.
\end{array}
$$

For $4{\cdot}2\acute{3}\times 6 = \frac{\overline{4{\cdot}23\times 10-4{\cdot}2}\times 6}{9}$ (Art. 22.) $= \frac{228{\cdot}6}{9}$ $= 25{\cdot}4$.

And $4{\cdot}2\acute{3}\times 6 = \overline{4{\cdot}2+\frac{{\cdot}3}{9}}\times 6$ (Art. 16.) $= 25{\cdot}2+{\cdot}2$.

For the Learner's Exerciſe.

Multiply $4{\cdot}\acute{2}$ by 7; $27{\cdot}\acute{6}$ by ·3; ${\cdot}10\acute{8}$ by ·04; $14{\cdot}\acute{8}$ by ·1 ${\cdot}27\acute{1}$ by ·005; ·9 by ${\cdot}31\acute{6}$.

2. A ſingle terminate, and a mixed multiple circulate.

Example.

Multiply $4{\cdot}2\acute{6}\acute{4}$ by ·3.

	Or thus,	
4·264		4·264,64, &c.
100		·3
426·4		1·2793,93, &c.
4·2		
422·2		
3		
1·2666		
1266		
12		
$1{\cdot}27\acute{9}\acute{3}$.		

For $4{\cdot}2\acute{6}\acute{4}\times{\cdot}3 = \frac{\overline{4{\cdot}264\times 100-4{\cdot}2}\times{\cdot}3}{99} = \frac{126{\cdot}66}{99}$ $= 1{\cdot}2666 + {\cdot}012666 + {\cdot}000126$, &c.

The other method is obvious.

For the Learner's Exerciſe.

Multiply $2{\cdot}\acute{4}1\acute{6}$ by 4; ${\cdot}8\acute{7}\acute{1}$ by ${\cdot}3$; ${\cdot}01\acute{4}\acute{6}$ by ${\cdot}7$; 2 by ${\cdot}\acute{4}\acute{7}$; ${\cdot}008$ by $3{\cdot}\acute{8}7\acute{1}$.

3. A terminate conſiſting of ſeveral figures, and a mixed ſingle circulate.

Example.

Multiply $2{\cdot}1\acute{6}$ by $4{\cdot}65$

$$\begin{array}{r} 2{\cdot}16\\ 10\\ \hline 21{\cdot}6\\ 2{\cdot}1\\ \hline 19{\cdot}5\\ 4{\cdot}65\\ \hline 9)90{\cdot}675\\ \hline 10{\cdot}075 \end{array}$$

Or thus,

$$\begin{array}{r} 4{\cdot}65\times2{\cdot}1\\ {\cdot}6\\ \hline 9)2{\cdot}790\\ \hline {\cdot}310\\ 465\\ 9\,30\\ \hline 10{\cdot}075 \end{array}$$

Or thus,

$$\begin{array}{r} 2{\cdot}166,\ \&c.\\ 4{\cdot}6\acute{5}\\ \hline 10\,8\acute{3}\\ 13\acute{0}\,00\\ 86\acute{6}\,66\\ \hline 10{\cdot}074\acute{9}. \end{array}$$

For $2{\cdot}1\acute{6}\times4{\cdot}65=\dfrac{\overline{2{\cdot}16\times10-2{\cdot}1}\times4{\cdot}65}{9}=\dfrac{90{\cdot}675}{9}$ $=10.075$.

And $2{\cdot}1\acute{6}=2{\cdot}1+{\cdot}0\acute{6}$ (Art. 16.) $=2{\cdot}1+\dfrac{{\cdot}6}{9}$, $\therefore$ $2{\cdot}1\times4{\cdot}65+\dfrac{{\cdot}6}{9}\times4.65=10{\cdot}075$.

The other proceſs is evident.

For the Learner's Practice.

Multiply $48{\cdot}\acute{7}$ by 874; $2{\cdot}8\acute{3}$ by ·21; $48{\cdot}\acute{2}$ by ·084; 48·76 by $21.\acute{8}$; ·016 by $\cdot 008\acute{7}$.

4. A terminate of ſeveral figures, and a mixed multiple circulate.

Example.

Multiply ·467 by $3\acute{2}{\cdot}\acute{4}$.

$$\begin{array}{r}32{\cdot}4\\100\\\hline 3240\\3\\\hline 3210\\467\\\hline 14{\cdot}9907\\1499\\14\\\hline 15{\cdot}14\acute{2}\acute{1}\end{array}$$

Or thus,

$$\begin{array}{r}32{\cdot}4{,}2\text{, \&c.}\\\cdot 467\\\hline 22\acute{6}\acute{9}\\19\acute{4}\acute{5}4\\12\acute{9}\acute{6}96\\\hline 15{\cdot}14\acute{2}\acute{1}\end{array}$$

For $3\acute{2}{\cdot}\acute{4} \times \cdot 467 = \dfrac{\overline{32{\cdot}4 \times 100 - 30 \times \cdot 467}}{99} = 14{\cdot}9907$ $+ \cdot 149907 + \cdot 00149$, &c.

The other method is plainly derived from Art. 19. 24.

For the Learner's Exerciſe.

Multiply 271·4 by $4{\cdot}\acute{8}7\acute{1}$; 271·6 by $1{\cdot}\acute{4}\acute{8}$; ·487 by $\cdot 2\acute{7}\acute{6}$; 27·14 by $4{\cdot}8\acute{4}1\acute{6}$; $\cdot 00\acute{1}\acute{9}$ by 2·14; $\cdot\acute{7}1\acute{4}$ by 4·876.

CASE III.

The factors, a pure circulate, and a mixed circulate.

RULE.

Multiply the mixed circulate by an *unit* with as many *cyphers* annexed as the repetend has places, and from the product ſubtract the terminate part; then multiply the remainder by the other repetend, having its decimal point removed as far to the right as it contains places, and reſerve the reſult. Write down as many *nines* as there are figures in the greater repetend, and annex as many *cyphers* as the places in the leſs repetend, from which ſubtract as many *nines*, and divide the reſerved reſult by the remainder; the quotient, continued till it terminates or repeats, is the true product. Or, inſtead of finding a diviſor, place the reſult under itſelf according to the places of either repetend as in Caſe I; and then place the ſum again under itſelf as before, obſerving the places of the other repetend; or divide it by as many *nines*.

1. A pure ſingle circulate, and a mixed ſingle circulate.

Example.

Multiply 2·3́ by ·7́.

```
        2·3          Or thus,       14·7
         10                         1 47
       ----                           14
        2 3                            1
          2                         ----
       ----                            ′
90       21                       9)16·3
 9        7                        -----
--     ----                       1·814,8,&c.
81 )147(1·814,8, &c.
    ···
```

For $\cdot\acute{7} \times 2\cdot\acute{3} = \frac{\cdot 7 \times 10}{9} \times \frac{2\cdot 3 \times 10 - 2}{9}$ (Art. 9.) $= \frac{147}{9 \times 9}$ $= \frac{147}{9 \times 10 - 9}$. (Art. 12.)

And $\frac{147}{9 \times 9} = 147 \times \frac{1}{9} \times \frac{1}{9} = \frac{14\cdot 7 + 1\cdot 47 + \cdot 147, \text{ \&c.}}{9}$

For the Learner's Exercise.

Multiply $\cdot\acute{4}$ by $2\cdot\acute{6}$; $\cdot\acute{8}$ by $\cdot 01\acute{4}$; $48\cdot\acute{7}$ by $\cdot\acute{4}$; $\acute{4}\cdot 6\acute{8}$ by $\acute{9}$; $\cdot 24\acute{6}$ by $\acute{2}$; $48\cdot\acute{7}$ by $\cdot 0\acute{4}$.

2. A pure ſingle circulate, and a mixed multiple circulate.

Example.

Multiply $4\acute{2}4\cdot\acute{3}$ by $\cdot 0\acute{3}$.

$$\begin{array}{r}
424\cdot 3 \\
1000 \\
\hline
424\ 300 \\
4 \\
\hline
423\ 900 \\
\cdot 3 \\
\hline
127\cdot 17 \\
127 \\
\hline
12\cdot 729 \\
1\ 272 \\
127 \\
12 \\
1 \\
\hline
1\acute{4}\cdot 1\ \acute{4}
\end{array}$$

Or thus,

$$\begin{array}{r}
4\acute{2}4\cdot\acute{3} \\
\cdot 3 \\
\hline
9)12\acute{7}\cdot 2\acute{9} \\
\hline
1\acute{4}\cdot 1\acute{4}
\end{array}$$

For $42\acute{4}\cdot\acute{3}\times\cdot0\acute{3}=\frac{424\cdot3\times1000-400}{999}\times\frac{\cdot3}{9}=\frac{127170}{9\times999}$ $=\overline{127\cdot17+\cdot12717+\cdot000127,\ \&c.}\times\frac{1}{9}=12\cdot729+1\cdot2729$ $+\cdot12729+\cdot0127$, &c.

The reaſon of the other proceſs is evident.

For the Learner's Exerciſe.

Multiply $\cdot2\acute{7}\acute{6}$ by $\cdot\acute{6}$; $4\acute{8}\cdot5\acute{1}$ by $\cdot\acute{3}$; $7\cdot1\acute{0}\acute{4}$ by $\cdot0\acute{2}$; $\cdot\acute{6}\acute{8}$ by $\cdot\acute{1}$; $\cdot0\acute{1}$ by $2\cdot\acute{4}6\acute{1}$; $\cdot0\acute{6}$ by $4\ \acute{0}0\acute{2}$.

3. A pure multiple circulate, and a mixed ſingle circulate.

Example.

Multiply $\acute{2}6\cdot\acute{8}$ by $\cdot02\acute{6}$.

$\cdot026$
10
$\cdot26$
$\cdot02$
$\cdot24$
26800
$6\cdot432$
$64.$
$9)6\cdot4\acute{3}\acute{8}$
$\cdot\acute{7}1538204\acute{8}$

Or thus,

$\acute{2}6\cdot\acute{8}\times\cdot02$
$\cdot06$
$9)1\cdot\acute{6}0\acute{9}$
$\cdot\acute{1}7884551\acute{2}$
$\cdot\acute{5}3\acute{6}$
$\cdot\acute{7}1538204\acute{8}.$

For $\acute{2}6\cdot\acute{8}\times\cdot02\acute{6} = \frac{26\cdot8\times1000}{999}\times\frac{\cdot026\times10-\cdot02}{9} = \frac{6432}{9\times999}$ $= \frac{6\cdot432+\cdot0064,\ \&c.}{9}$.

And $\cdot02\acute{6} = \cdot02+\cdot00\acute{6} = \cdot02+\frac{\cdot06}{9}$ (Art. 16. 7.) $\therefore\ \cdot02\acute{6}\times\acute{2}6\cdot\acute{8}$ $= \cdot02\times\acute{2}6\cdot\acute{8}+\frac{\cdot06\times\acute{2}6\cdot\acute{8}}{9}$.

For the Learner's Practice.

Multiply $\acute{4}\cdot6\acute{8}$ by $4\cdot\acute{1}$; $\cdot\acute{4}\acute{1}$ by $6\cdot8\acute{7}$; $\cdot\acute{6}8\acute{1}$ by $\cdot004\acute{6}$; $\cdot4\acute{6}$ by $\acute{2}\cdot4\acute{6}$; $\cdot27\acute{8}$ by $\acute{4}\cdot\acute{6}$.

4. A pure multiple circulate, and a mixed multiple circulate.

Example.

Multiply $\cdot04\acute{2}\acute{3}$ by $\cdot0\acute{3}\acute{8}$.

$$\begin{array}{rr} & \cdot0423 \\ & 100 \\ \hline & 4\cdot23 \\ & \cdot04 \\ \hline 9900 & 4\cdot19 \\ 99 & 3\cdot8 \\ \hline 980\ 1 &)\underset{\cdot\cdot}{15}\cdot\underset{\cdot\cdot\cdot}{922}(\cdot001624528+ \end{array}$$

Or thus,

$$\begin{array}{r} \cdot04\acute{2}\acute{3} \\ 3\cdot\dot{8} \\ \hline 33\acute{8}\ \acute{5} \\ 12\acute{6}\acute{9} \\ \hline \cdot00160\acute{8}\acute{2},\ 82,\ 82,\ \&c. \\ 160\ \ 82,\ 82,\ \&c. \\ 1\ \ 60,\ 82,\ \&c. \\ 1\ \ 60,\ \&c. \\ \hline \cdot0016245\ 28\ + \end{array}$$

For $\cdot 04\acute{2}\acute{3} \times \cdot 0\acute{3}\acute{8} = \frac{\cdot 0423 \times 100 - \cdot 04}{99} \times \frac{3\cdot 8}{99} = \frac{15\cdot 922}{99 \times 100 - 99}$.

And $\cdot 0\acute{3}\acute{8} = 3\cdot 8 \times \frac{1}{99} \therefore \cdot 04\acute{2}\acute{3} \times 3\cdot 8 \times \frac{1}{99} = \cdot 001608\acute{8}\acute{2} +$

$\cdot 00001608\acute{8}\acute{2}$ + &c.

For the Learner's Exerciſe.

Multiply $\acute{2}4\cdot\acute{8}$ by $6\cdot\acute{8}\acute{4}$; $\acute{6}8\cdot\acute{7}$ by $4\acute{6}\cdot\acute{8}$; $\acute{2}1\cdot 8\acute{2}$ by $4\cdot\acute{6}\acute{8}$; $\acute{6}82\cdot\acute{1}$ by $\acute{4}\cdot\acute{6}$; $2\cdot\acute{1}\acute{9}$ by $\cdot\acute{4}1\acute{6}$.

CASE IV.

When the factors are both pure circulates.

RULE.

Multiply the numbers as terminates, and remove the decimal point as many places to the right-hand as there are figures in both repetends; then find a diviſor, agreeable to the places of the repetend; or, add the products as before.

1. Two pure ſingle circulates.

Example.

Multiply $\cdot\acute{8}$ by $\cdot 0\acute{3}$.

$\cdot 8$
$90\ \cdot 03$
9 —
—
$81)2\cdot 4(\cdot 0\acute{2}9\acute{6}$

Or thus,

$\cdot\acute{8}$
$\cdot 3$
—
$9)\cdot 2\acute{6}$
—
$\cdot 0\acute{2}9\acute{6}$.

For $\cdot\acute{8} \times \cdot 0\acute{3} = \frac{\cdot 8 \times 10}{9} \times \frac{\cdot 03 \times 10}{9} = \frac{\cdot 8 \times \cdot 03 \times 100}{9 \times 9}$.

And $\cdot 0\acute{3} = \frac{\cdot 3}{9} \therefore \cdot\acute{8} \times \cdot 0\acute{3} = \frac{\cdot\acute{8} \times \cdot 3}{9}$.

For the Learner's Exerciſe.

Multiply $\cdot\acute{8}$ by $\cdot\acute{4}$; $\cdot 0\acute{7}$ by $\cdot 0\acute{1}$; $\cdot\acute{4}$ by $\cdot\acute{2}$; $\cdot 06$ by $\cdot\acute{1}$; $\cdot 00\acute{4}$ by $\cdot\acute{5}$.

2. A pure ſingle circulate, and a pure multiple circulate.

Example.

Multiply $421\cdot\acute{4}2\acute{1}$ by $\cdot 00\acute{2}$.

$$\begin{array}{l} 421\cdot \\ \cdot 002 \\ \hline 8\cdot 42 \\ \quad\;\; 84 \text{ \&c.} \\ \hline \end{array}$$

$$9)\acute{8}\cdot 4\acute{2}$$

$$\overline{\cdot 93649\acute{2}04760\acute{3}.}$$

For $421\cdot\acute{4}2\acute{1} \times \cdot 00\acute{2} = \frac{421 \times 1000}{999} \times \frac{\cdot 002 \times 10}{9} = \frac{8420 \times \frac{1}{999}}{9}$
$= \frac{8\cdot 42 + \cdot 00842, \text{ \&c.}}{9}$

For the Learner's Practice.

Multiply $\acute{6}8\cdot\acute{2}$ by $\cdot\acute{2}$; $\acute{1}7\cdot\acute{6}$ by $\cdot\acute{8}$; $.\acute{7}6\acute{2}$ by $\cdot\acute{0}\acute{4}$; $\cdot\acute{6}\acute{8}$ by $\cdot 00\acute{6}$; $\cdot 0\acute{4}$ by $\cdot\acute{6}8\acute{3}$.

3. Two pure multiple circulates.

Example.

Multiply $\cdot\acute{3}2\acute{4}$ by $\acute{4}\cdot\acute{6}$.

$$\begin{array}{r} \cdot 324 \\ 4\cdot 6 \\ \hline 149\cdot 04 \\ 149\,.\,. \\ \hline 1\cdot 49\acute{1}8\acute{9}1 \\ 14918 \\ 149 \\ 1 \\ \hline \acute{1}\cdot 5069\acute{6} \end{array}$$

Or thus,

$$\begin{array}{r} \cdot\acute{3}2\acute{4} \\ 460 \\ \hline 19\acute{4}\acute{5} \\ 1\acute{2}9\acute{7} \\ \hline 99)14\acute{9}\cdot 1\acute{8} \\ \hline \acute{1}\cdot 5069\acute{6}. \end{array}$$

$$\text{For } \cdot\acute{3}2\acute{4} \times \acute{4}\cdot\acute{6} = \frac{\cdot 324 \times 1000}{999} \times \frac{4\cdot 6 \times 100}{99} = \cdot 324 \times 4\cdot 6 \times 100000 \times \frac{1}{999} \times \frac{1}{99} = 1\cdot 4\acute{9}1\acute{8} + \cdot 014\acute{9}1\acute{8} + \cdot 00014, \text{ \&c.}$$

$$\text{And } \acute{4}\cdot\acute{6} = \frac{460}{99} \therefore \frac{\cdot\acute{3}2\acute{4} \times 460}{99} = \acute{1}\cdot 5069\acute{6}.$$

For the Learner's Exercise.

Multiply $\cdot\acute{8}7\acute{1}$ by $\acute{4}\cdot\acute{1}$; $\acute{6}8\cdot\acute{2}$ by $\cdot\acute{0}1\acute{4}$; $\cdot\acute{6}0\acute{1}$ by $\cdot\acute{0}0\acute{2}$; $\cdot\acute{4}\acute{8}$ by $\cdot\acute{0}4\acute{8}$.

CASE

CASE V.

If both factors are mixed circulates.

RULE.

Multiply the numbers as terminates, and remove the decimal point as many places to the right hand as the number of figures in both repetends; to which add the product of the terminate parts, and reſerve the ſum. Multiply the terminate parts by the circulates reciprocally, having their decimal points brought forward according to their places, and ſubtract the ſum of theſe products from the reſerved ſum: then proceed as before, by finding a diviſor; or adding the remainders.

1. Two mixed ſingle circulates.

Example.

Multiply $2{\cdot}1\acute{4}$ by $4{\cdot}\acute{3}$.

$$\begin{array}{r}
2{\cdot}14\\
4{\cdot}3\\ \hline
920{\cdot}2\\
4\times2{\cdot}1=\ 8{\cdot}4\\ \hline
928{\cdot}6\\
175{\cdot}9\\ \hline
9)\,752{\cdot}7\\ \hline
9)\ \ 83{\cdot}63\\ \hline
9{\cdot}2\acute{9}2\acute{5}
\end{array}
\qquad
\begin{array}{r}
43\\
2{\cdot}1\\ \hline
90{\cdot}3
\end{array}
\qquad
\begin{array}{r}
21{\cdot}4\\
4\\ \hline
85{\cdot}6
\end{array}$$

$$175{\cdot}9 = 90{\cdot}3 + 85{\cdot}6$$

Or thus,

$$\begin{array}{r}
2{\cdot}1\acute{4}\times4\\
3\\ \hline
9)\,6{\cdot}4\acute{3}\\ \hline
{\cdot}\acute{7}14\acute{8}\\
8{\cdot}\acute{5}777\\ \hline
9{\cdot}2\acute{9}2\acute{5}
\end{array}$$

For $2{\cdot}1\acute{4}\times4{\cdot}\acute{3}=\frac{2\cdot14\times10-2\cdot1}{9}\times\frac{4\cdot3\times10-4}{9}=$

$$\frac{2{\cdot}14\times4{\cdot}3\times100+2{\cdot}1\times4-\overline{2{\cdot}1\times4{\cdot}3\times10+4\times2{\cdot}1\times10}}{9\times9}.$$

And $4\cdot\acute{3} = 4 + \frac{3}{9} \therefore 2\cdot\acute{1}\acute{4} \times 4\cdot\acute{3} = 4 \times 2\cdot\acute{1}\acute{4} + \frac{3 \times 2\cdot\acute{1}\acute{4}}{9}$.

For the Learner's Exerciſe.

Multiply $2\cdot\acute{8}$ by $41\cdot\acute{6}$; $6\cdot8\acute{7}$ by $\cdot02\acute{1}$; $\cdot6\acute{1}$ by $4\cdot8\acute{7}$; $4\cdot8\acute{9}$ by $6.\acute{4}$; $\cdot4\acute{6}$ by $\cdot01\acute{7}$.

2. A mixed ſingle circulate, and a mixed multiple circulate.

Example.

Multiply $\cdot01\acute{7}$ by $1\cdot\acute{0}0\acute{3}$.

$$\begin{array}{r}
1\cdot003 \\
\cdot017 \\
\hline
170\cdot51 \\
\cdot01 \\
\hline
170\cdot52 \\
10\cdot2 \\
\hline
\cdot16032 \\
160 \\
\hline
9)\cdot160\acute{4}\acute{8} \\
\hline
\cdot01\acute{7}8311644\acute{9}
\end{array}$$

Or thus,

$$\begin{array}{r}
1\cdot\acute{0}0\acute{3} \times \cdot01 \\
\cdot07 \\
\hline
9)\cdot07\acute{0}2\acute{1} \\
\hline
\cdot00780113446 \text{ \&c.} \\
\cdot01003003003 \\
\hline
\cdot01\acute{7}8311644\acute{9}
\end{array}$$

For $\cdot01\acute{7} \times 1\cdot00\acute{3} = \frac{\cdot017 \times 10 - \cdot01}{9} \times \frac{1\cdot003 \times 1000 - 1}{999} = \frac{\overline{170\cdot51 + \cdot01 - 10\cdot2} \times \frac{1}{999}}{9}$.

And $\cdot01\acute{7} = \cdot01 + \frac{\cdot07}{9}$.

For the Learner's Exercise.

Multiply $46\cdot\acute{8}$ by $4\acute{6}\cdot\acute{8}$; $\cdot 21\acute{8}$ by $4\cdot\acute{6}\acute{8}$; $26\cdot\acute{4}$ by $\cdot 9\acute{0}2\acute{4}$; $7\acute{1}\cdot\acute{6}$ by $\cdot 4\acute{6}$; $4\acute{8}\cdot 9\acute{1}$ by $6\cdot 82\acute{7}$.

3. Two mixed multiple circulates.

Example.

Multiply $\cdot 01\acute{3}\acute{2}$ by $2\acute{0}\cdot\acute{1}$.

$$\begin{array}{rr}
 & \cdot 0132 \\
 & 20\cdot 1 \\ \cline{2-2}
 & 2653\cdot 2 \\
 & \cdot 2 \\ \cline{2-2}
9900 & 2653\cdot 4 \\
99 & 46\cdot 5 \\ \hline
9801 &)2606\cdot 9(\cdot 26598306 + \\
 & \cdot\;\cdot\;\cdot\;\cdot\;\cdot
\end{array}
\qquad \text{Or thus,} \qquad
\begin{array}{r}
2\acute{0}\cdot\acute{1}\times\cdot 01 \\
\cdot 32 \\ \hline
\cdot 064\acute{3}\acute{2},32,32, \\
64,32,32, \\
64,32 \\
64 \\ \hline
\cdot 06497296 \\
\cdot 20101010 \\ \hline
\cdot 26598306 +
\end{array}$$

For $\cdot 01\acute{3}\acute{2}\times 2\acute{0}\cdot\acute{1}=\dfrac{\cdot 0132\times 100-\cdot 01}{99}\times\dfrac{20\cdot 1\times 100-20}{99}$ $=\dfrac{2606\cdot 9}{99\times 100-99}$.

And $\cdot 01\acute{3}\acute{2}=\cdot 01+\cdot 00\acute{3}\acute{2}=\cdot 01+\dfrac{32}{99}$.

For the Learner's Exercise.

Multiply $1\acute{4}\cdot 6\acute{8}$ by $2\cdot\acute{6}\acute{8}$; $\cdot 1\acute{6}\acute{8}$ by $\cdot 01\acute{4}\acute{6}$; $6\acute{5}7\cdot 1\acute{2}$ by $\cdot 4\acute{8}7\acute{1}$; $\cdot 6\acute{8}71\cdot 9\acute{6}$ by $68\acute{7}19\cdot\acute{5}$.

SECTION

SECTION V.

Diviſion of Circulates.

CASE I.

When the dividend is a circulate (pure or mixed) and the diviſor a terminate.

RULE.

PROCEED as in common diviſion, continuing the repeating figures in the dividend (if neceſſary) till the quotient repeats or terminates.

Example 1.

Divide $\cdot\acute{4}2\acute{7}$ by 8.

$8)\cdot427,427,4$ &c.$(.053\acute{4}2\acute{8},4$ &c.

Example 2.

Divide $2\cdot1\acute{6}8\acute{4}$ by $68\cdot7$.

$68\cdot7)2\cdot1684,684,6$ &c.$(\cdot0315641$ &c.

Examples for the Learner's Exerciſe.

Divide $\acute{6}\cdot8\acute{7}$ by 27; $\cdot2\acute{4}\acute{1}$ by $8\cdot1$; $68\cdot\acute{4}\acute{1}$ by $\cdot091$; $\acute{6}\cdot4\acute{1}$ by $\cdot06$; $\cdot02\acute{1}\acute{4}$ by 48.

CASE

CASE II.

When the dividend is a terminate, and the diviſor a pure circulate.

RULE.

Multiply the terminate by as many *nines* as there are repeating figures in the circulate for a new dividend, and remove the decimal point in the circulate the ſame places to the right for a new diviſor; then proceed as in common diviſion, continuing the quotient (if neceſſary) till it repeats or terminates.

Example.

Divide 7 by $\cdot\acute{6}$.

$$\begin{array}{r} 7 \\ \underline{9} \\ 6)63(10\cdot5 \end{array}$$

For $\cdot\acute{6} = \frac{6}{9}$, and $7 \div \frac{6}{9} = \frac{9}{6} \times \frac{7}{1} = \frac{63}{6} = 10\cdot5$.

For the Learner's Practice.

Divide 4 by $\cdot\acute{6}$; $\cdot8$ by $2\cdot\acute{2}$; $\cdot6$ by $\cdot0\acute{8}$; $\cdot5$ by $\cdot0\acute{8}$; $\cdot06$ by $\cdot\acute{4}$; $\cdot007$ by $\acute{2}$.

2. A terminate conſiſting of ſeveral figures, and a pure ſingle circulate.

Example.

Divide 43·2 by $\cdot\acute{7}$.

$$\begin{array}{r} 43\cdot2 \\ \underline{9} \\ 7)388\cdot8(55\cdot\acute{5}42857\acute{1} \end{array}$$

For $\cdot\acute{7} = \frac{7}{9}$, and $43\cdot2 \div \frac{7}{9} = \frac{9}{7} \times \frac{43\cdot2}{1} = \frac{388\cdot8}{7}$ $\doteq 55\cdot54$, &c.

For the Learner's Practice.

Divide 764 by $\cdot\acute{3}$; $\cdot68$ by $8\cdot\acute{8}$; $\cdot2816$ by $\cdot00\acute{6}$; $2\cdot2$ by $\cdot00\acute{7}$; $\cdot087$ by $\cdot\acute{5}$.

3. A ſingle terminate, and a pure multiple circulate.

Example.

Divide $\cdot6$ by $\acute{2}\cdot3\acute{1}$.

$$\begin{array}{r} \cdot6 \\ 999 \\ \hline \end{array}$$

2310)599·4(·25948, &c.

For $\acute{2}\cdot3\acute{1} = \frac{2310}{999}$, and $\cdot6 \div \frac{2310}{999} = \frac{999}{2310} \times \cdot6$ $= \cdot25948$, &c.

For the Learner's Exerciſe.

Divide 8 by $\acute{2}3\cdot\acute{1}$; $\cdot7$ by $\acute{8}7\cdot1\acute{6}$; $\cdot08$ by $\cdot0\acute{8}\acute{7}$; 4 by $619\cdot\acute{6}1\acute{9}$; $\cdot009$ by $\cdot00\acute{1}\acute{7}$.

4. A terminate of ſeveral figures, and a pure multiple circulate.

Example.

Divide $48\cdot76$ by $\acute{3}2\cdot4\acute{1}$.

$$\begin{array}{r} 487600 \\ 48\cdot76 \\ \hline \end{array}$$

324100)487551·24(1·5043, &c.

For $\dot{3}2\cdot4\dot{1} = \frac{324100}{9999}$, and $48\cdot76 \div \frac{324100}{9999} = \frac{9999 \times 48\cdot76}{324100}$ $= \frac{48\cdot76 \times 10000 - 48\cdot76}{324100}$.

For the Learner's Exercise.

Divide, 3871 by $\dot{4}6\cdot\dot{8}$; 48·71 by $\dot{7}1\cdot\dot{6}$; ·0087 by $\dot{2}\cdot68\dot{7}$; 487·1 by $\cdot\dot{6}87\dot{1}$; $271.\dot{2}\dot{7}1$.

CASE III.

If the dividend be a terminate, and the diviſor a mixed circulate.

RULE.

Multiply the terminate by as many *nines* as there are places in the repetend for a new dividend; then remove the decimal point in the circulate the ſame places to the right-hand, from which ſubtract the given circulate, and the remainder will be the new diviſor, with which proceed as in common diviſion.

1. A ſingle terminate, and a mixed ſingle circulate.

Example.

Divide 8 by $3\cdot2\dot{7}$.

$$\begin{array}{rl}
3\cdot27 & \\
32\cdot77 & 8 \\
3\cdot27 & 9 \\
\hline
29\cdot5 &)72(2\cdot44067,\ \&c.
\end{array}$$

For $3\cdot2\dot{7} = \frac{3\cdot2\dot{7} \times 10 - 3\cdot27}{9} = \frac{29\cdot5}{9}$, and $8 \div \frac{29\cdot5}{9} = \frac{9}{29\cdot5} \times 8 = \frac{72}{29\cdot5}$.

For the Learner's Practice.

Divide, 7 by $48{\cdot}\acute{7}$; ${\cdot}4$ by $48{\cdot}\acute{2}$; ${\cdot}02$ by $2{\cdot}\acute{7}$; ${\cdot}008$ by ${\cdot}08\acute{7}$; ${\cdot}00\acute{7}$ by $6{\cdot}\acute{8}\acute{7}$; 8 by $46.\acute{9}$.

2. A ſingle terminate, and a mixed multiple circulate.

Example.

Divide ${\cdot}6$ by $3{\cdot}2\acute{7}\acute{4}$.

$$\begin{array}{lr} 3{\cdot}2\acute{7}\acute{4} & \\ 327{\cdot}474\ \&c. & 6 \\ 3{\cdot}274\ \&c. & 99 \\ \hline 324{\cdot}2 &)59{\cdot}4({\cdot}18321\ \&c. \end{array}$$

For $3{\cdot}2\acute{7}\acute{4} = \frac{324{\cdot}2}{99}$, $\therefore {\cdot}6 \div \frac{324{\cdot}2}{99} = \frac{99 \times 6}{324{\cdot}2}$.

For the Learner's Practice.

Divide 4 by $2{\cdot}1\acute{6}\acute{8}$; ${\cdot}8$ by $6\acute{6}{\cdot}8\acute{1}$; ${\cdot}\acute{7}$ by ${\cdot}4\acute{8}\acute{7}$; 6 by ${\cdot}2\acute{1}\acute{9}$; ${\cdot}04$ by $5\acute{8}1{\cdot}90\acute{1}$.

3. A terminate of ſeveral figures, and a mixed ſingle circulate.

Example.

Divide $4{\cdot}56$ by $3{\cdot}1\acute{4}$.

$$\begin{array}{lr} 3{\cdot}14 & \\ 31{\cdot}4 & 4{\cdot}56 \\ 3{\cdot}1 & 9 \\ \hline 28{\cdot}3 &)41{\cdot}04(1{\cdot}45017,\ \&c. \end{array}$$

For $3{\cdot}1\acute{4} = \frac{28{\cdot}3}{9}$, $\because$ $4{\cdot}56 \div \frac{28{\cdot}3}{9} = \frac{4{\cdot}56 \times 9}{28{\cdot}3}$.

For the Learner's Practice.

Divide, 768 by $2{\cdot}1\acute{4}$; $2{\cdot}07$ by $2{\cdot}8\acute{7}$; ${\cdot}0681$ by $21{\cdot}4\acute{6}$; ${\cdot}716$ by ${\cdot}41\acute{6}$; ${\cdot}0081$ by ${\cdot}087\acute{4}$.

4. A terminate of ſeveral figures, and a mixed multiple circulate.

Example.

Divide $13{\cdot}271$ by $4\acute{7}{\cdot}68\acute{4}$.

```
47·684
47 6847·6    132710.
     47·6      13·271
-----------  ---------
476800.     )132696·729(·2783, &c.
```

For $4\acute{7}{\cdot}68\acute{4} = \frac{476800}{9999}$, $\because$ $13{\cdot}2171 \div \frac{476800}{9999} = \frac{9999 \times 13{\cdot}271}{476800} = \frac{13{\cdot}271 \times 10000 - 13{\cdot}271}{476800}$.

For the Learner's Exerciſe.

Divide, $2{\cdot}714$ by $28{\cdot}\acute{4}1\acute{6}$; $3{\cdot}871$ by $\acute{4}{\cdot}87\acute{1}$; ${\cdot}0487$ by $4{\cdot}\acute{8}\acute{1}$; ${\cdot}487$ by ${\cdot}9\acute{1}6\acute{8}$; $46{\cdot}87$ by $2\acute{1}{\cdot}0\acute{1}$.

CASE IV.

If the dividend be a pure circulate, and the diviſor a mixed circulate.

RULE.

Remove the decimal point of the dividend as many places to the right-hand as repeating figures in both numbers, from which ſubtract the ſaid dividend, punctuated according to its place, for a new dividend. Then multiply the diviſor by an an *unit* with as many *cyphers* annexed as its repetend hath places; this product leſſened by its terminate part, and the decimal point removed to the right, according to the places of the firſt dividend, and again leſſened by the laſt remainder, will give the new diviſor.

1. A pure ſingle circulate, and a mixed ſingle circulate.

Example.

Divide $\cdot\acute{6}$ by $2\cdot\acute{3}$.

$$\begin{array}{rl} 2\cdot3 & \\ 10 & \\ \hline 23 & \\ 2 & \\ \hline 21 & \cdot\acute{6} \\ 210 & 60 \\ 21 & 6 \\ \hline 189\,) & 54(\cdot28571, \&c. \end{array}$$

Or thus, $21)6.(\cdot28571$, &c.

For $\cdot\acute{6} = \frac{6}{9}$, and $2\cdot\acute{3} = \frac{2\cdot\acute{3}\times10-2\cdot\acute{3}}{9} = \frac{21}{9}$, $\therefore \frac{6}{9} \div \frac{21}{9} =$

$$\frac{9}{21} \times \frac{6}{9} = \frac{6\times10-6}{2\cdot3\times10-2\times10-2\cdot3\times10-2} = \frac{6}{21}.$$

For the Learner's Exercise.

Divide $\cdot\acute{4}$ by $2\cdot\acute{6}$; $\cdot\acute{8}$ by $4\acute{6}\cdot\acute{8}$; $\cdot0\acute{8}$ by $\cdot68\acute{7}$; $\cdot\acute{7}$ by $68\cdot4\acute{1}$; $\cdot0\acute{8}$ by $46\cdot9\acute{8}$.

2. A pure ſingle circulate, and mixed multiple circulate,

Example.

Divide $\cdot0\acute{7}$ by $6\acute{3}1\cdot\acute{4}$.

$$\begin{array}{rcl}
6\acute{3}1\cdot\acute{4} & & \\
100 & & \\
\cline{1-1}
63140 & & \\
6 & & \\
\cline{1-1}
62540 & & \cdot07 \\
625400 & & 700 \\
62540 & & \cdot7 \\
\cline{1-1}\cline{3-3}
562860 &) & 699\cdot3(\cdot00124 \text{ \&c.}
\end{array}$$

For $\cdot0\acute{7} \div 6\acute{3}1\cdot\acute{4} = \frac{\cdot0\acute{7}\times10000-\cdot07\times10}{6\acute{3}1\cdot\acute{4}\times100-6\times10-6\acute{3}1\cdot\acute{4}\times100-6}$

$= \frac{699\cdot3}{562860}$.

For the Learner's Practice.

Divide $\cdot\acute{4}$ by $27\cdot\acute{1}\acute{4}$; $7\cdot\acute{7}$ by $\cdot4\acute{8}7\acute{1}$; $\cdot00\acute{8}$ by $38\cdot\acute{2}\acute{6}$; $6\cdot\acute{6}$ by $\cdot2\acute{1}\acute{6}$; $11\cdot\acute{1}$ by $4\cdot\acute{8}18\acute{7}$.

3. A pure multiple circulate, and a mixed ſingle circulate.

Example.

Divide $\acute{3}4\cdot\acute{6}$ by $\cdot04\acute{3}$.

$$\begin{array}{ll}
\cdot04\acute{3} & \\
10 & \\
\hline
\cdot43 & \\
\cdot04 & \\
\hline
\cdot39 & \acute{3}4\cdot\acute{6} \\
390 & 346000 \\
\cdot39 & 34600 \\
\hline
389\cdot61 &)\,311400(799\cdot26079, \&c.
\end{array}$$

For $\acute{3}4\cdot\acute{6}\div\cdot04\acute{3}=\dfrac{\acute{3}4\cdot\acute{6}\times10000-34\cdot6\times1000}{\overline{\cdot043\times10-\cdot04}\times1000-\overline{\cdot043\times10-\cdot04}}$

$=\dfrac{311400}{389\cdot61}$.

For the Learner's Practice.

Divide $\acute{2}\cdot4\acute{1}$ by $71\cdot\acute{4}$; $\acute{2}\cdot4\acute{8}$ by $\cdot27\acute{1}$; $\acute{4}1\cdot6\acute{8}$ by $\cdot06\acute{4}$; $\cdot\acute{4}8\acute{7}$ by $68\cdot\acute{9}$; $\acute{4}6\cdot\acute{8}$ by $48\cdot\acute{8}$.

4. A pure multiple circulate, and a mixed multiple circulate.

Example.

Example.

Divide $\cdot 0\acute{4}\acute{2}$ by $.03\acute{2}\acute{6}$.

$$\begin{array}{llr}
\cdot 03\acute{2}\acute{6} & & \text{Or thus,}\quad 3\cdot 23)4\cdot 2(1\cdot 3003,\ \&c, \\
\underline{\quad 100} & & \\
3\cdot 26 & & \\
\underline{\ \cdot 03} & & \\
3\cdot 23 & \cdot 042 & \\
3\,23 & 420 & \\
\underline{\quad 3\cdot 23} & \underline{\quad 4\cdot 2} & \\
319\cdot 77\ ) & 415\cdot 8(1\cdot 3003,\ \&c. &
\end{array}$$

For $\cdot 0\acute{4}\acute{2} \div \cdot 03\acute{2}\acute{6} = \frac{\cdot 042 \times 100}{99} \div \frac{\cdot 0326 \times 100 - \cdot 03}{99} = \frac{4\cdot 2}{3\cdot 23}$.

For the Learner's Exerciſe.

Divide $\acute{4}\cdot\acute{2}6$ by $42\cdot\acute{7}\acute{6}$; $4\acute{1}\cdot\acute{6}$ by $\cdot 07\acute{1}\acute{4}$; $\cdot 0\acute{8}\acute{7}$ by $\cdot 005\acute{6}\acute{8}$; $\acute{2}6\cdot\acute{8}$ by $4\cdot\acute{1}\acute{7}$; $\acute{6}0\cdot 0\acute{1}$ by $.017\acute{5}\acute{6}$.

Remark 1.

It appears from the firſt and laſt Examples, that when the places of repeating figures in both numbers are equal, the work may be contracted, by rejecting the denominators in the finite values of the given circulates.

2. It is alſo evident that this rule holds good *(mutatis mutandis)* when the *dividend* is a mixed circulate, and the *diviſor* a pure circulate.

Example.

Example.

Divide $4\cdot20\acute{7}\acute{6}$ by $\acute{3}4\cdot1\acute{2}$.

$$\begin{array}{rr}
 & 4\cdot20\acute{7}\acute{6} \\
 & 100 \\
\hline
 & 420\cdot76 \\
 & 4\cdot2 \\
\hline
34\cdot12 & 416\cdot56 \\
34\,120000 & 416\,5600 \\
341200 & 416\cdot56 \\
\hline
33778800 &)\,4165183\cdot44(\cdot1233,\ \&c.
\end{array}$$

$$\text{For } 4\cdot20\acute{7}\acute{6}\div\acute{3}4\cdot1\acute{2}=\frac{\overline{4\cdot2076\times100-4\cdot20}\times10000-\overline{4\cdot2076\times100-4\cdot2}}{34\cdot12\times1000000-34\cdot12\times1000.}.$$

For the Learner's Practice.

Divide $4\cdot\acute{2}\acute{7}$ by $\acute{2}\cdot1\acute{6}$; $8\acute{4}\cdot7\acute{1}$ by $\acute{5}01\cdot\acute{6}$; $4\cdot\acute{8}7\acute{1}$ by $\acute{4}0\cdot1\acute{7}$; $\cdot6\acute{2}1\acute{0}$ by $\acute{5}\,07\acute{6}$; $\cdot001\acute{7}\acute{6}$ by $\cdot0\acute{1}4\acute{6}$; $\cdot007\acute{4}\acute{3}$ by $\cdot00\acute{1}71\acute{6}$.

CASE V.

When both the dividend and diviſor are pure circulates.

RULE.

Find a new dividend and a new diviſor, by removing the decimal points as many places to the right-hand as there are repeating figures in both numbers, and then ſubtracting the numerators of their reſpective terminate values.

1. Two pure ſingle circulates.

Example.

Divide ·6́ by ·04́.

```
·04́  ·6́        Or,      ·4)6(15
 4   60
 ·4   6
———  ——
3·6 )54(15.
```

$$\text{For } \cdot\acute{6} = \frac{6}{9}, \text{ and } \cdot 0\acute{4} = \frac{\cdot 4}{9}, \therefore \frac{6}{9} \div \frac{\cdot 4}{9} = \frac{\cdot 6 \times 100 - 6}{\cdot 04 \times 100 - 4} = \frac{6}{\cdot 4}.$$

For the Learner's Practice.

Divide ·7́ by ·8́; ·04́ by 2·2́; 7·7́ by ·6́; ·01́ by 4·4́; ·8́ by ·07́; ·008́ by ·0004́.

2. Two pure multiple circulates.

Example.

Divide ·4́32́ by 2́·3́.

```
2·3          ·432
2 30000      43200
    230        432
———————      —————
229770 )4276 (·18526, &c.
```

For the Learner's Exerciſe.

Divide 4́6·8́ by 6́·7́; ·5́04́ by 1́·68́; ·0́14́ by 7́·65́; 2́1·8́ by ·001́4́; ·0́819́ by ·0́71́.

3. A pure ſingle circulate, and a pure multiple circulate.

Example.

Example 1.

Divide $\cdot 00\acute{4}$ by $341\cdot\acute{3}4\acute{1}$

$$\begin{array}{r l}
341 & \cdot 004 \\
3410000 & \quad 40 \\
341000 & \qquad \cdot 04 \\
\hline
3069000 \;) & 39\cdot 96(\cdot 00001302, \&c.
\end{array}$$

Example 2.

Divide $\acute{7}0\cdot 04\acute{6}$ by $4\cdot\acute{4}$.

$$\begin{array}{r l}
4\cdot 4 & \\
4\,000000 & 70046000 \\
40 & \;7004600 \\
\hline
3999960 &)63041400(15\cdot 7605, \&c.
\end{array}$$

For the Learner's Practice.

Divide $\cdot\acute{4}$ by $\cdot\acute{6}8\acute{2}$; $\cdot 0\acute{8}$ by $\cdot\acute{4}2\acute{7}$; $\cdot 00\acute{1}$ by $5\cdot\acute{0}\acute{4}$; $\cdot 0\acute{8}7\acute{1}$ by $\cdot\acute{7}$; $\acute{4}\cdot 8\acute{1}$ by $\cdot 0\acute{4}$.

CASE VI.

If the dividend and diviſor are both mixed circulates.

RULE.

Find the new diviſor and dividend by removing the decimal point of the terminate value of each numerator as many places to the right-hand as there are repeating figures in the other number, and then ſubtracting their reſpective numerators.

1. Two mixed ſingle circulates.

Example.

Divide $21\cdot\acute{4}$ by $3\cdot\acute{4}$.

$$\begin{array}{ll} 3\cdot4 & 21\cdot\acute{4} \\ 34 & 21\,4 \\ 3 & 21 \\ \hline 310 & 1930 \\ 31 & 193 \\ \hline 279 &)1737(6\cdot2258, \text{\&c.} \end{array}$$

Or thus, 31(193(6·2258, &c.

For $21\cdot\acute{4} \div 3\cdot\acute{4} = \frac{9}{3\cdot4\times10-3} \times \frac{21\cdot4\times10-21}{9} =$ $\frac{193\times10-193}{31\times10-31} = \frac{193}{31}$.

For the Learner's Practice.

Divide $41\cdot\acute{7}$ by $8\cdot1\acute{7}$; $98\cdot\acute{4}$ by $\cdot41\acute{8}$; $\cdot12\acute{7}$ by $\cdot04\acute{6}$; $1\cdot0\acute{4}$ by $\cdot08\acute{7}$; $\cdot71\acute{4}$ by $\cdot008\acute{4}$; $\cdot68\acute{7}$ by $99\cdot\acute{9}$.

2. Two mixed multiple circulates.

Example.

Divide $\cdot01\acute{3}\acute{2}$ by $3\acute{2}\cdot0\acute{1}$.

$$\begin{array}{ll} 31980 & 1\cdot31 \\ 3198000 & 1\,310 \\ 31980 & 1\cdot31 \\ \hline 3\acute{1}66020 &)1308\cdot69(\cdot0004133, \text{\&c.} \end{array}$$

For the Learner's Exerciſe.

Divide $4{\cdot}8\acute{7}\acute{1}$ by $\cdot6\acute{1}\acute{8}$; $4{\cdot}\acute{1}0\acute{9}$ by $\cdot7\acute{1}8\acute{1}$; $\cdot4\acute{2}7\acute{1}$ by $\cdot07\acute{1}\acute{4}$; $5{\cdot}\acute{0}7\acute{1}$ by $\cdot007\acute{1}\acute{4}$; $\cdot4\acute{8}7\acute{6}$ by $\cdot01\acute{4}3\acute{6}$; $\cdot09\acute{8}71\acute{4}$ by $4\acute{8}{\cdot}7165\acute{9}$.

3. A mixed ſingle circulate and a mixed multiple circulate,

Example 1.

Divide $\cdot01\acute{6}$ by $1{\cdot}\acute{0}1\acute{4}$.

1013 ·15
10130 150.
1013 ·15

9117)149·85(·01643, &c.

Example 2.

Divide $21{\cdot}\acute{0}\acute{7}$ by $4{\cdot}2\acute{1}$.

37·9 21050
37 900 210500
37·9 21050

37862·1)189450(5·0034, &c.

For the Learner's Exerciſe.

Divide $4{\cdot}1\acute{6}$ by $2{\cdot}\acute{1}8\acute{4}$; $21{\cdot}8\acute{7}$ by $4{\cdot}1\acute{0}\acute{6}$; $187{\cdot}1\acute{9}$ by $\cdot01\acute{7}\acute{6}$; $\cdot4\acute{8}7\acute{1}$ by $50{\cdot}\acute{1}$; $2\acute{0}1{\cdot}4\acute{7}$ by $\cdot407\acute{1}$,

SECTION VI.

Of the Logarithms of Repeating Decimals.

HAVING, in the foregoing Sections, ſupplied the pupil with rules for managing the whole doctrine of circulating numbers by common Arithmetic, and given the theory and reaſons for the ſame, we ſhall therefore now proceed to ſhew how the whole buſineſs may be eaſily performed *logarithmically*, having firſt premiſed the following

LEMMA.

When one number is to be divided by another, the quotient will be the ſame as if *unity* were divided by the latter number, and the quotient multiplied by the former. For $\frac{A}{B} = \frac{1 \times A}{B} = \frac{1}{B} \times A$.

From hence, and the nature of Logarithms, it is evident, that $L.\frac{A}{B} = L.A - L.B = L.A + \overline{L.1 - L.B}$; and ſince in the terminate value of any circulating number, the denominator conſiſts of as many *nines* as figures in the repetend, we ſhall evidently have this general rule for the logarithm of any circulate, pure or mixed.

RULE.

To the log. of the numerator of the terminate value of the given circulate add the arithmetical complement of as many *nines* as the repetend has places, and the ſum will be the logarithm of the given circulate.

Example

Example 1.

Required the logarithm of the repetend $\cdot\acute{2}$.

$$\cdot\acute{2} = \frac{2}{9}, \therefore$$

To the tabular logarithm of 2 - - - = 0·3010300
Add the arithmetical complement of the logarithm of 9 - = ·9·0457574
The sum is the logarithm of the circulate $\cdot\acute{2}$ = ·9·3467874

Example 2.

What is the logarithm of $\acute{4}\cdot6\acute{4}$?

$$\acute{4}\cdot6\acute{4} = \frac{4640}{999}, \therefore$$

To the log. of 4640 - - - - - - - = 3·6665180
Add the arithmetical complement of 999 - = ·7·0004344
The sum is the logarithm of - $\acute{4}\cdot6\acute{4}$ - = 0·6669524

Example 3.

Required the logarithm of $\cdot00\acute{7}14\acute{5}$.

$$\cdot00\acute{7}14\acute{5} = \frac{71\cdot45}{9999}, \therefore$$

To the log. of - - - - - - 71·45 - = 1·8540022
Add the arith. comp. of - - - 9999 - = ·6·0000433
The sum is the logarithm of - $\cdot00\acute{7}14\acute{5}$ = ·7·8540455

Example

Example 4.

Required the logarithm of $4\cdot\acute{3}$.

$$4\cdot\acute{3} = \frac{39}{9}, \therefore$$

To the log. of - - - - - - - 39 - = 1·5910646
Add the arith. comp. of - - - 9 - = ·9·0457574
The sum is the logarithm of - - $4\cdot\acute{3}$ - = 0·6358220

Example 5.

What is the logarithm of $2\acute{3}5\cdot\acute{7}$?

$$2\acute{3}5\cdot\acute{7} = \frac{235500}{999}, \therefore$$

To the log. of - - - - 235500 - = 5·3719909
Add the arith. comp. of - - - 999 - = ·7·0004344
The sum is the logarithm of - $2\acute{3}5\cdot\acute{7}$ - = 2·3724253

For the Learner's Exercise.

Required the logarithms of the following repeating decimals.

$21\cdot\acute{4}$; $7\acute{6}\cdot\acute{8}$; $\cdot\acute{0}76\acute{4}$; $46\cdot\acute{6}$; $2\acute{8}\cdot1\acute{9}$; $466\cdot\acute{9}$; $\cdot\acute{1}8\acute{7}$; $68\cdot21\acute{4}$; $\acute{4}\cdot68\acute{7}$; $71\acute{6}\cdot49\acute{8}$; $24\cdot\acute{2}\acute{4}$; $\cdot00\acute{0}8\acute{7}$; $\cdot6\acute{9}\acute{8}$; $18\acute{7}\cdot164\acute{9}$; $487\acute{1}6\cdot871\acute{4}$; $2\acute{8}76\cdot814\acute{3}$; $\cdot00000\acute{6}\acute{7}$.

A Table of the nine Digits perpetually circulating, and of the Arithmetical Complements of the Denominators.

Number.		Logarithm.	Number.		Logarithm.
$\dot{1}$	=	0·0457574	$\frac{1}{9}$	=	·9·0457574
$\dot{2}$	=	0·3467874	$\frac{1}{99}$	=	·8·0043647
$\dot{3}$	=	0·5228786	$\frac{1}{999}$	=	·7·0004344
$\dot{4}$	=	0·6478174	$\frac{1}{9999}$	=	·6·0000433
$\dot{5}$	=	0·7447274	$\frac{1}{99999}$	=	·5·0000043
$\dot{6}$	=	0·8239086	$\frac{1}{999999}$	=	·4·0000004
$\dot{7}$	=	0·8908554	$\frac{1}{9999999}$	=	·3·0000000
$\dot{8}$	=	0·9488474			
$\dot{9}$	=	1·0000000			

PRACTICAL QUESTIONS,

ADAPTED TO THE

PRECEDING RULES.

Queſtion I.

WHAT is the Product of 14 feet 8 inches, by 11 feet 10 inches?

That is $14\cdot\acute{6} \times 11\cdot8\acute{3} = \frac{106\cdot5}{9} \times \frac{132}{9} = 173\cdot\acute{5} = 173$ f. 6 i. 8 p.

Queſtion II.

Required the Area of 18 f. 9 i. 10 p. by 11 i. 9 p.?

That is $\frac{169\cdot375}{9} \times \frac{8\cdot8125}{9} =$ 18 f. 5 i. 1 p. 6 th. 6 fo.

Queſtion III.

There is a Hall, the length of which is 96 f. 7 $\frac{2}{3}$ i. and the breadth 68 f. 8 $\frac{1}{3}$ i. which is to be paved with marble ſquares, each 1 f. 1 $\frac{1}{2}$ i. ſqu. How many ſuch ſquares will it take, and what does it come to at 2s 6d $\frac{1}{2}$ *per* foot?

Queſtion IV.

If 1 C^{wt}. 2 q^{rs}. 18 lb. of Sugar coſt 5l 1s 8d. what will 12 C^{wt}. 3 q^{rs}. coſt at the ſame rate?

Queſtion V.

What muſt I give for 8 lb of Tobacco, when ½ C^{wt}. coſt 4l 17s 8d?

Queſtion VI.

If the Penny-loaf weigh 7 ¾ Ounces, when Wheat is at 6s 4d *per* Buſhel, what muſt be the weight of the Penny-loaf when Wheat is at 3s 10d *per* Buſhel?

Queſtion VII.

Suppoſe 4 Hh^{ds}, 3 Firkins, and 5 Gallons of Beer, coſt 6l 14s 8d. How much is that *per* Hh^{d}, and *per* G^{n}.?

Queſtion VIII.

A piece of Land 4 rods broad and 40 long, being a Statute-Acre; it is required to know what length, with 10 rods and 2 yards breadth, will make an Acre?

Queſtion IX.

A Bill of Exchange was accepted at London for the Payment of £847·53́, for the ſame value delivered at Liſbon in Millrees; Exchange at 5s 4d *per* piece. How many Millrees were paid at Liſbon?

Queſtion X.

A Merchant is deſirous to know how much of each of the following Wines he muſt take, ſo that the whole Quantity may be 84 ⅝ G^{ns}. at 5s 10d *per* G^{n}. Malaga at 7s 6d. Canary at 6s 9d. Sherry at 5s. and White Wine at 4s 3d. *per* G^{n}.

Queſtion XI.

In what time will £7 *per Annum* pay a Debt of 120l 8s at 6 *per Cent.* Simple Intereſt?

Queſtion XII.

Suppoſe a Freehold Eſtate of £40 $\frac{2}{3}$ *per Ann.* is to be ſold; what is it worth, allowing the Buyer £5 *per Cent.* Comp. Int. for his Money?

Queſtion XIII.

What is the Diſcount of 87l 13s 4d for 233 days, at £4 *per Cent. per Ann.*?

Queſtion XIV.

A challenges B to run a Race with him, if he will give him 3 rods in 10; now the velocity of B's running to that of A, is as 31 to 22. Which of the two beat?

Queſtion XV.

In the Year of our Lord 1775, the Cycle of the Sun was 9, and the Cycle of the Moon 20. Required from hence the Year of the Dionyſian Period.

Queſtion XVI.

Required the Area of the Parabola, whoſe Ordinate and Abſciſſa are $60{\cdot}\acute{4}\acute{2}$ and $\acute{4}1{\cdot}2\acute{4}$ reſpectively.

Queſtion XVII.

Required the Solidity of a Spheroid, of which the greater Axis is 12 $\frac{2}{3}$, and the leſs $6{\cdot}4\acute{2}\acute{7}$.

Queſtion XVIII.

Required the Quadrature of Hippocrates' Lunes, C. D. (Fig. 1.) the Diameter A B being 18 $\frac{1}{6}$, and A E 14 $\frac{2}{3}$.

Queſtion XIX.

What is the Solidity of a Parabolic Conoid, the Diameter of the Baſe being $\overset{\prime}{9}{\cdot}7\overset{\prime}{5}$, and the Height $11{\cdot}2\overset{\prime}{5}$?

Queſtion XX.

How many Solid Inches are contained in an Icoſahedron, the ſide of which is $2{\cdot}\overset{\prime}{4}$ Inches?

MISCELLANEOUS QUESTIONS.

Queſtion XXI.

"There are five whole numbers, the three firſt of which are in geometrical progreſſion, and the three laſt in arithmetical, the ſecond being the common difference in the laſt three. Now the ſum of the four laſt $= 102$, and the product of the ſecond and laſt number is 504. Required the numbers."

Anſwer. Let v, w, x, y, z, repreſent the five numbers; then, *per Queſt.* $vx = w^2$, $x + w = y$, $x + 2w = z$, $w + x + y + z = 102$, and $wz = 504$. For y and z ſubſtitute their values in the 4th. Equat. and it becomes $4w + 3x = 102$. And from the 3d and 5th Equat. we find $x = \frac{504}{w} - 2w$; this value of x being ſubſtituted in the laſt expreſſion, gives $4w + \frac{1512}{w} - 6w = 102$, reduced, $w^2 + 51w = 756$, hence $w = 12$, and the required numbers are 8, 12, 18, 30, and 42.

Queſtion XXII.

"To find the leaſt whole number, which, being divided by 19. ſhall leave 17; if divided by 28, ſhall leave 21 remainder; and if the ſum of the two reſulting quotients be ſubtracted from the number ſought, the remainder, being divided by 3, ſhall leave 2 remaining."

Anſwer. The leaſt whole number that can poſſibly ſatisfy the two firſt conditions of the Queſtion is 245. *per* Simpſon's Algebra, *p.* 289. From whence likewiſe it is plain that the number ſought may be repreſented by $532x + 245$. But in order to obtain a general expreſſion for the ſum of the reſulting quotients, we muſt obſerve, that as 532 is a multiple of the Diviſors 19 and 28, any multiple (x) of the ſaid number will likewiſe contain a like multiple of theſe diviſors; and 245 being conſtant, the quotients 12 and 8 will be ſo too. Therefore $\overline{19+28} \times x + 20$ will truly repreſent the ſum of the quotients, which being ſubtracted from the number ſought leaves $485x+225$. Now 2 being taken from this number, and the remainder divided by 3, the quotient $\frac{485x+223}{3}$ will be a whole number by the queſtion, from whence the leaſt value of x will be found $= 1$, and conſequently that of $532x+245 = 777$; which is the number required.

Note. As $47x+20$ is a general expreſſion for the ſum of the quotients, any other diviſor and remainder beſides 3 and 2 may be propoſed, and the number anſwering the conditions of the queſtion found as above.

Queſtion XXIII.

" Required the dimenſions of the greateſt cylinder that can be inſcribed in a ſolid, formed by the rotation of a curve round its axis, whoſe equation is $ax^2 - x = y^4$, abſciſs 40, and ſemiordinate 28 inches?"

Anſwer. The greateſt cylinder that can be inſcribed in a ſolid, generated by a curve revolving about its axis, is when the height of the cylinder is equal to half the ſubtangent (*per* Emerſon's Conics, *p.* 53). From the equation of the curve we have $\frac{4ax^2-4x}{2ax-1}$ = the ſubtangent; therefore $\frac{4ax^2-4x}{4ax-2}$ $= 40-x$, reduced $x = \sqrt{\left.\overline{\frac{160a+6}{16a}}\right|^{2} - \frac{10}{a}} + \frac{160a+6}{16a} =$ 19·9988. Therefore 40—19·9988 = 20·0011 = the height of the greateſt inſcribed cylinder; from whence the diameter is eaſily found by the equation of the curve.

Note. Had the equation of the curve been $ax^2-2x=y^4$, then the expreſſion for the ſubtangent $\frac{4ax^2-4x}{2ax-2}$ would be evidently $= 2x$ (which is the property of the common or Apollonian Parabola) and the height of the cylinder = 20 exactly.

Queſtion XXIV.

"If in a plane triangle a right line be drawn from the vertical angle to the baſe, forming an angle at the ſame equal to the complement of half that at the vertex; the line ſo drawn will divide the difference of the ſegments of the baſe, in the ratio of the ſides, including the vertical angle: required the Demonſtration."

Anſwer. Draw BD (fig. B) making AD = the difference of the ſegments; and make BEC = the complement of half the vertical angle ABC. Let BF be taken = BC; and CF be drawn, and likewiſe AH ∥ to DB. Then it is evident becauſe of parallels, that the angle HAB = ABD = BCA—BAC = the Difference of the angles at the baſe. And in the triangles EGC, FGB, the angle FGB = EGC; and BED = BFG *per* conſtruc. therefore GCE = FBG. But (*per* Simpſon's Trig. *p.* 62.) the angle GCE = half the difference of the angles at the baſe; conſequently BE biſects the angle ABD, whence AB : BD (= BC) :: AE : ED (*per* Euc. 3, 6.) *Q.E.D.*

Queſtion XXV.

"In a plane triangle ABC are given two ſides, AC, BC, and the line CD = BC drawn from the vertex C to terminate in and biſect the baſe AB, to conſtruct the triangle geometrically."

Anſwer. Let CA (fig. C.) = one of the given ſides; with the other given ſide CB, deſcribe BD; from A, draw the tangent AE, on which deſcribe the ſemicircle AFE; with AF = FE deſcribe FD, draw ADB and it is done. For $AF^2 + FE^2 = 2\,AF^2 = AE^2 = AD \times AB$; but $AD = AF$ *per* conſtruc. $\because AD \times AB = AF \times \overline{AF + DB} = AF^2 + AF \times DB$; hence $2\,AF^2 = AF^2 + AF \times DB$, conſequently $AF\ (AD) = DB$. *Q.E.D.*

Queſtion XXVI.

"In lat. 53° N. ſtands a Tower the ſhade of whoſe ſummit on Tueſday *June* 9, 1772, deſcribed a curve on the plane of the horizon whoſe tranverſe axis was 150 yards: required the height of the ſaid Tower geometrically."

Anſwer. It is too well known to need demonſtrating here, that when the ſun's declination is leſs than the complement of latitude, the curve formed by the ſection of the horizontal plane with the Cone of Rays is an hyperbola. Therefore on the given line AB (fig. D.) = the tranſverſe axis, deſcribe the ſegment of a circle to contain an angle of twice the ſun's given declination. Make the angle BAD = the difference of the colatitude and the ſaid declination; and draw DC ⊥ AE, and it will be the height of the tower required. For let DE biſect the angle BDF, and AG be perpend. to ED produced; then will the angle GDA = EDF = the codeclin. hence the ∠ GAD = the ſun's declination. And becauſe GE is evidently parallel to the earth's axis, the angle GAE = the colatitude; therefore the ∠ DAB = the difference of the colat. and the given declination. Moreover becauſe BDF is the cone of rays deſcribed by the ſun in his parallel, whoſe vertex is D, and AC the plane of the horizon cutting it in the hyperbola BH, AB is manifeſtly the tranſverſe axis; conſequently DC is the true height of the ſummit from the horizontal plane. The calculation is very eaſy; for in the triangle ADB, we have AB and all the angles to find another ſide, ſuppoſe AD; from whence, and the angle DAC, we find DC = 43.688.

N. B. I have taken the ſun's declination = 23°.

Queſtion XXVII.

" The ſhorteſt ſide of a right-angled triangle is given, and a perpendicular let fall from the right-angle cuts the hypothenuſe in extreme and mean proportion. It is required to conſtruct the triangle geometrically."

Anſwer. On the line AB (fig. E) drawn at pleaſure, make AD = the given ſide of the triangle; raiſe the indefinite ⊥ DC, and take BD to AD in the given proportion. Then make BC = DA, and join CA, and ABC will be the required triangle.

For BA is divided in the given ratio by the ⊥ DC, and BC = the given ſide *per* conſtruc. and ſince BA × BD ($=DA^2$) = BC^2, the ∠ at C is a right one, *per* Simpſon's Geom. *p*. 4. Theo. 19.

Queſtion

Queſtion XXVIII.

"In the common experiment of the double cone rolling (apparently) upwards, how far will it move between the indefinite right lines, which ſupport it, ſuppoſing they meet in an angle of 12 degrees, and the plane of ſituation elevated 4 degrees above the horizon, the common baſe of the double cone being 6 inches and diſtance between the vertices 13 inches?"

Anſwer. It is well known, in the experiment of the double cone moving apparently upwards, that the center of gravity thereof does actually deſcend; hence, ſhould the right lines or rulers which ſupport the cone be ſo poſited, as to cauſe the center of gravity (on its being rolled upwards) either to move horizontally, or to recede from the baſe or horizontal line, it manifeſtly can have no motion but what is cauſed by an external power.

It alſo appears from experiment, that the cone will move with a greater or leſs celerity, according as the right lines or rulers are inclined to one another in a greater or leſs angle, and that the angle may be made ſo acute, or the rulers brought ſo near together, as to cauſe the cone to roll downwards; conſequently there muſt be a certain angle, in which the cone will reſt in any part of the inclined plane —— now this angle is very eaſily determined as follows: Let ABCD (fig. F.) be an horizontal plane, and AIHD a plane inclined to it in the angle FMG. Let alſo MH, MI, repreſent the indefinite right lines or rulers, and KMLE a vertical ſection of the double cone.

Then becauſe FG is parallel and equal to NM, and HI parallel and equal to KL, it is evident, that while the plane

KMLE ſlides vertically betwixt the lines MH, MI, (which is the ſame thing as the cone rolling) the point N will move horizontally on the line NF, and when it arrives at F, the lines KL, HI, will coincide. And moreover, from this equality and parallelism of the lines, we have this Analogy, as NM:NL, or as EM : KL :: FG : FI. But if we call FM radius, FG will be the ſine of the angle FMG, and FI the tangent of the angle FMI; therefore as the diameter of the common baſe of two cones, is to the diſtance of their vertices, ſo is the the ſine of the plane's elevation above the horizon, to the tangent of half the angle included by the two right lines or rulers. Hence, ſhould the given angle be leſs than that found by this proportion, it is plain, that the cone cannot aſcend, ſince on its being rolled upwards, the center of gravity muſt move upwards likewiſe. Let this now be applied to the preſent caſe, and we ſhall find that the angle, or inclination, of the indefinite right lines, ought to be ſomething more than 17° 12′, to cauſe the leaſt apparent aſcent of the cone; but this exceeds the given angle in the Queſtion (viz. 12°) conſequently the cone can in this caſe have no motion of itſelf. This will appear ſtill plainer, if we ſuppoſe the cone to be rolled along the inclined plane, to any given diſtance (ſuppoſe 10 inches) from the point of contact of the rulers; for its center of gravity in that ſituation would be found by calculation to have receded from the horizontal baſe line 6·212 inches, and conſequently the center of gravity would have aſcended ·212 of an inch, which is contrary to both theory and experiment.

We might purſue this matter ſtill farther, by ſhewing the greateſt angle of elevation of the inclined plane, as well as the greateſt or leaſt dimenſions of the cone, whereby it could poſſibly aſcend. But as all this muſt be very obvious to any one who duly attends to the principle of gravitation, it is needleſs to ſay any thing more.

Queſtion XXIX.

Required a general rule for the inſcribing of regular Polygons in a given circle.

The rule given by Mr. Malton in his Royal Road to Geometry, Prob. 25. which is alſo the ſame with that in Ward's Mathematician's Guide, Prob. 20, is this,

" Draw a diameter AB, on which conſtruct an equilateral triangle ADB; or draw the two arks only, interſecting at D, (this preparation is the ſame for any polygon whatever;) then divide the diameter into as many equal parts as the polygon, required, has ſides; and through the ſecond diviſion, from either extreme, draw a right line from D to the oppoſite ſide of the concave circumference." Upon this Prob. Mr. Malton makes the following remarks:

" Thus may a ſide of any polygon whatever, contained in a circle, be obtained; by obſerving the rule given above. And it is truly worthy of notice, that any right line drawn from D, cutting the diameter and the concave circumference, will cut them both in the ſame proportion; or in whatever ratio one of them is divided, a right line being drawn, from D, through the point of diviſion, will alſo cut the other in the ſame ratio."

" Of this conſtruction or equal diviſion of the diameter and the circumference, no demonſtration can be given, having conſulted ſeveral able Geometricians concerning it; who ſay, that it is only an approximation, and not mathematically true.

Yet I muſt own, that I do believe it to be perfectly true, or it could never anſwer ſo very accurately, as it does, in all diviſions whatever."

It ſeems to me ſomething very extraordinary to ſee a profeſſed Geometrician reaſon ſo very *ungeometrically*. I always thought that not even a mere reader, much leſs a reformer of Euclid, could give his aſſent to the truth of a geometrical conſtruction barely from a ſeeming concurrence of points, or coincidence of lines; but from an obvious regular deduction from firſt principles. For I am very clear, that there can be nothing effected by lines (at leaſt in plane Geometry) but a demonſtration may be given, directly or indirectly, of its truth or falſity. If Geometry were founded on no better a baſis than the bare teſtimony of external ſenſe, I am afraid we ſhould ſoon view the whole fabric in ruins. Mr. Malton, through his whole performance, ſeems to lay a great ſtreſs on an *ocular* demonſtration. From whence it ſhould ſeem, that in order to become a proficient in geometry, it is indiſpenſably neceſſary to be furniſhed with the whole apparatus of a good microſcope, which ſhould be the criterion of every linear conſtruction. It is to be hoped, however, that the following inveſtigation will fully convince this Gentleman (without relying wholly on our *optic* faculty) that this rule is ſo far from being "perfectly true" for all regular polygons, that it anſwers in one caſe only, when the coſine of the angle at the center ſubtended by the ſide of the polygon is equal to half the radius, which is eaſily ſhewn to be the property of an arch of 60 or 120 degrees, anſwering to the trigon or hexagon.

It is ſomewhat ſurprizing that ſo many able Mathematicians ſhould be conſulted, in order to be ſatisfied of the truth or falſity of this rule, which may be ſo eaſily demonſtrated in the following manner.

Biſect AB (fig. G.) the diam. of the given circle in C, through which perpendicular to AB, draw indefinitely HD. Take AI to AB as AG is to AGB, and draw GID. Then muſt D evidently be the true point, from whence a right line being drawn through the given point I, will divide the diam. and concave circumference in the ſame ratio. Draw GF perpendicular to AB; then will CD be a fourth proportional to IF, IC, and FG. But FG is the ſine, and FC equal the coſine of the arch AG, and CI is known from AB and AI being given; hence we have this analogy for finding the point D, ſo that DG ſhall divide AB and AHB in the ſame ratio. As the coſine of AG—CI : ſine of AG :: CI : CD. Thus for the trigon, if we call AC, 1, AI will be equal $\frac{2}{3}$, and thence IC equal $\frac{1}{3}$ ∴ As Coſ. 60°—$\frac{1}{3}$: S. 60° :: $\frac{1}{3}$: 1·73205 =CD. After the ſame manner we find the diſtance CD for the regular polygons as follow:

Pentagon, as Coſ. 72°—$\frac{1}{5}$: S. 72° :: $\frac{1}{5}$: 1·74478, =CD.

Hexagon, as Coſ. 60°—$\frac{1}{3}$: S. 60° :: $\frac{1}{3}$: 1·73205, =CD.

Heptagon, as Coſ. 51°, 21′$\frac{5}{7}$—$\frac{3}{7}$: S. 51°, 21′$\frac{5}{7}$ :: $\frac{3}{7}$: 1·71903, = CD.

Octagon, as Coſ. 45°—$\frac{1}{2}$: S. 45° :: $\frac{1}{2}$: 1·707106, =CD.

Nonagon, as Coſ. 40°—$\frac{5}{9}$: S. 40° :: $\frac{5}{9}$: 1·69654, =CD.

Decagon, as Coſ. 36°—$\frac{3}{5}$: S. 36° :: $\frac{3}{5}$: 1·68728, =CD.

Undecagon, as Coſ. 32°, 43′$\frac{7}{11}$ − $\frac{7}{11}$: S. 32°, 43′$\frac{7}{11}$ :: $\frac{7}{11}$: 1·679165 = CD.

Duodecagon, as Coſ. 30°—$\frac{2}{3}$: S. 30° :: $\frac{2}{3}$: 1·67202, =CD.

Now, by Malton's (Ward's) rule CD is equal $\sqrt{\overline{AD}|^2-\overline{AC}|^2}$ $= \sqrt{3} = 1{\cdot}73205$; which correſponds only with the trigon or hexagon. The reaſon why this conſtruction anſwers to the arch of 60 or 120° will be evident if we conſider, that in this caſe, CD is double of GF, and thence FI is $\frac{1}{3}$ of FC; but FC is $= \frac{1}{2}$, and therefore AI is $= \frac{1}{3}$ of AB; which is the ſame part as the arch AG is of AHB. Hence it plainly appears, that the point D found by Ward's method is not true for any polygon whatever, excepting in one ſingle caſe; for it is evident that the ſide of the tetragon cannot be ſaid to be found by this conſtruction; for by the above analogy it will be, as Coſ. 90°—1 : S. 90° :: 1 : $\frac{1}{0}$, which ſhews that CD in this caſe is infinite.

The latter part of Mr. Malton's 47th Prob. (which is alſo Ward's) is in the ſame predicament with the other, being proved to be falſe as follows:

The conſtruction being made according to the rule (fig. H.) it is evident from the like ſituation of the circles, &c. that wherever the point B is taken in GK the lines EF, DC, will be always parallel to each other, and the ∠ DCF = CDE, as alſo DEF = CFE. The ſides CF, FE, and ED, are likewiſe equal, being each a radius of the ſame or equal circles. What remains then to be proved is, whether the ∠ DCF be = the greater angle in a regular pentagon, formed by the diagonal and a ſide. The moſt direct (if not the only) method of inveſtigating which appears to be by a calculation of that angle.

Having drawn ſuch lines as appear by the fig. EF will be the ſide of a regular hexagon, and FB the ſide of a regular dodecagon, inſcribed in the ſame circle, and they are therefore in the ratio of 1 to $\sqrt{2-\sqrt{3}}$. And ſince FG is ⊥ IK, and KC ∥ FG, the ∠ GKC is a right ∠; and the ∠ KBC = ABI is evidently $= \frac{1}{2}$ a right ∠. Alſo, as BF is the ſide of a dodecagon, the ∠ FIB $= \frac{1}{3}$ of a right ∠, and conſequently the ∠FBI $= \frac{5}{6}$ of a right ∠; but the ∠ CBF is the ſup. of KBC + FBI ∵ the ∠ FBC $= \frac{2}{3}$ of a right ∠ = 60°. Then as the ſines of the ∠s FBC, FCB are in the ratio of their oppoſite ſides FC, FB, we have, as $1 : \sqrt{2-\sqrt{3}}$ :: S. 60° : S. 26°, 38′, 2″ nearly; hence the ∠DCF = DCB + BCF = 71°, 38′, 2″. But the ∠ DCF in a regular pentagon is 72°. Therefore this conſtruction is falſe alſo.

N. B. Since the above was written, the inveſtigation of the veracity of theſe rules has been propoſed in the Ladies Diary, to which the above anſwers were ſent with ſome alterations.

Queſtion XXX.

To find the ſum of a ſeries conſiſting of 1000 ſurſolid numbers whoſe roots are $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, &c.

Anſwer. The general expreſſion for the ſum of a ſeries of powers as $2^n + 3^n + 4^n$, &c. to $\overline{r-1}$ terms is $\frac{r^{n+1}}{n+1} + \frac{r^n}{2} + \frac{nr^{n-1}}{3\cdot4}$, &c. $-\frac{1}{n+1} - \frac{1}{2} - \frac{n}{3\cdot4} + \frac{n+\overline{n-1}\times\overline{n-2}}{2\cdot3\cdot4\cdot5\cdot6.}$, &c. And ſince the roots of the propoſed ſeries are evidently in Arith. Progreſs. and the common difference equal to the firſt term, we may eaſily adapt this general equation to the given ſeries by adding unity to both ſides thereof, and multiplying the whole by $\left.\overline{\frac{1}{6}}\right|^5$. For then it becomes $\left.\overline{\frac{1}{6}}\right|^5 + \left.\overline{\frac{2}{6}}\right|^5 + \left.\overline{\frac{3}{6}}\right|^5 - - - - + \left.\overline{\frac{r}{6}}\right|^5 \Big(= \left.\overline{\frac{1}{6}}\right|^5 + \left.\overline{\frac{1}{3}}\right|^5 + \left.\overline{\frac{1}{2}}\right|^5 - - - - - + \left.\overline{\frac{r}{6}}\right|^5$ the given ſeries$\Big) = \left.\overline{\frac{1}{6}}\right|^5 \times : \frac{r^{n+1}}{n+1} + \frac{r^n}{2} + \frac{nr^{n-1}}{3.4} - \frac{n \times \overline{n-1} \times \overline{n-2}\, r^{n-3}}{2.3.4.5.6.}$, &c. $-\frac{1}{n+1} + \frac{1}{2} - \frac{n}{3.4}$, &c. (till it terminates) $= \left.\overline{\frac{1}{6}}\right|^5 \times : \frac{\left.\overline{1000}\right|^6}{6} + \frac{\left.\overline{1000}\right|^5}{2} + \frac{5 \times \left.\overline{1000}\right|^4}{12} - \frac{\left.\overline{1000}\right|^2}{12}$ $= 21497824502732{\cdot}767$ the ſum required.

Queſtion XXXI.

In a geometrical progreſſion beginning from unity, having the common ratio and number of terms given, the ſum of all the changes in the ſeries will be expreſſed by this general theorem $\frac{\overline{r^n-1} \times 1.2.3 - - - - - n}{br-b} \times : 1 + 10 + 100 + 1000$, &c. - - - - - to b terms, where each term muſt be diſtinctly conſidered, and eſtimated according to its variable local value in the ſeries. Required the demonſtration or inveſtigation?

Queſtion XXXII.

If A be any given number, P the places of figures, and C the number of changes, then will the ſum of all thoſe changes be repreſented by this ſeries $\frac{AC}{P} + \frac{10AC}{P} + \frac{100AC}{P}$, &c. - - - - - continued to P terms. Required the inveſtigation.

Queſtion XXXIII.

Required to find a fraction ſuch, that being taken from its reciprocal the remainder ſhall be a ſquare.

Fig. B

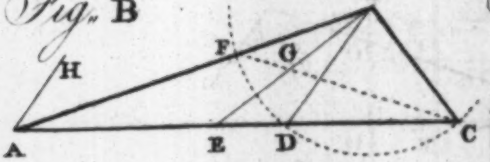


Fig. C

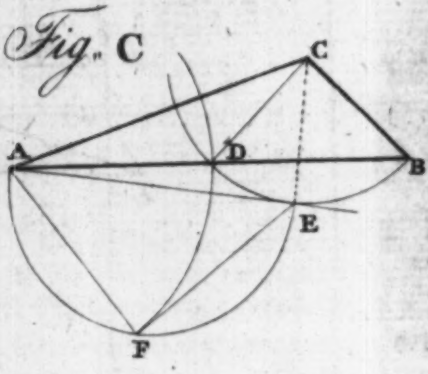


Fig. D

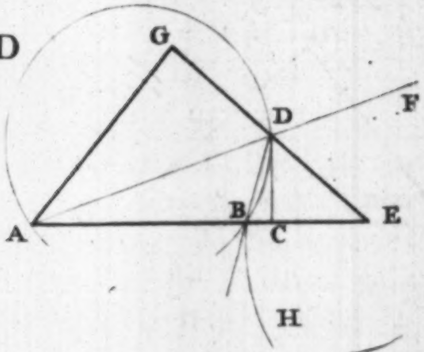


Fig. E

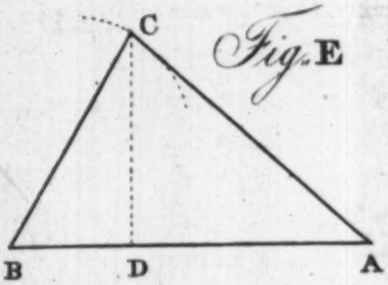


Fig. F

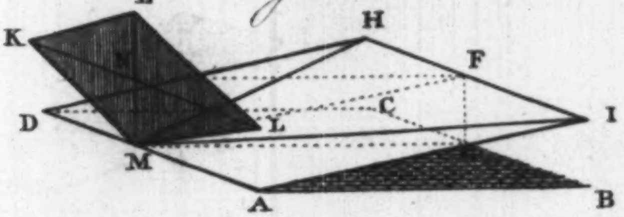


Fig. G

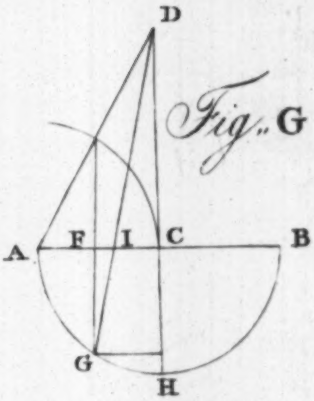
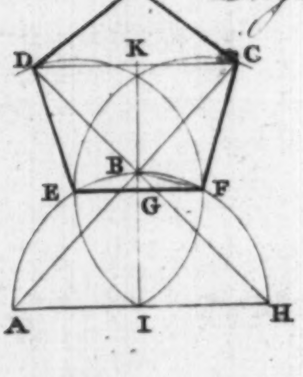


Fig. H



This queſtion was propoſed in the Ladies' Diary for 1768; but as the poſſibility, or impoſſibility, of ſuch a fraction exiſting has never yet been ſhewn, and therefore the queſtion not anſwered, I thought it not improper to do it here.

If we put $\frac{y}{x}$ for the required fraction; then, *per* queſtion, $\frac{x}{y} - \frac{y}{x}$ is to be a ſquare number; but $\frac{x}{y} - \frac{y}{x}$ is equal $\frac{x^2 - y^2}{xy}$, therefore the queſtion is, To find two numbers ſuch, that their product and the difference of their ſquares may be ſquare numbers. In order then to inveſtigate two ſuch numbers, if poſſible, put $xy = r^2$, then will $x = \frac{r^2}{y}$; therefore $\frac{r^4}{y^2} - y^2 = \square$ *per* queſt. Now it is evident that $\frac{r^2}{y}$ will be in the ſame ratio to y, as the hyp. of a right-angled triangle is to a ſide; that is, as $h : s :: \frac{r^2}{y} : y$, $\because y = \sqrt{\frac{s}{h}} \times r$. From whence it appears that if a right-angled triangle can be found, of which the hyp. and either ſide are rational ſquare numbers, the queſt. may be anſwered, otherwiſe it is impoſſible. But that it is abſolutely impoſſible for ſuch a triangle to be formed, will be very evident from the general expreſſions for the ſides of a commenſurate right-angled triangle, *viz.* $m^2 + n^2$, $m^2 - n^2$, and $2mn$. For ſince $m^2 + n^2$, and $m^2 - n^2$ (or $2mn$) are both to be rational ſquare numbers, it is plain that in the former expreſſion m and n muſt themſelves repreſent the two legs of a triangle; and in the latter m muſt alſo repreſent the hyp. and n a leg of a triangle; hence it follows, that either m muſt at the ſame time have two different values, or that there muſt be a right-angled triangle exiſting the hyp. of which is equal to one of its ſides; both which concluſions are abſurd.

The abſurdity of this queſtion may be alſo ſhewn rather differently, thus; Since $x^2 - y^2$ will be always a rational ſquare number when $x = \frac{y^2+r^2}{2r}$, and xy will be always a rational ſquare when $x = \frac{r^2}{y}$, (where y and r may be taken at pleaſure) it appears, that if $\frac{x^2-y^2}{xy}$, be a ſquare number, $\frac{y^2+r^2}{2r}$ muſt neceſſarily be equal to $\frac{r^2}{y}$. Therefore if we ſuppoſe r of the ſame value in both expreſſions, (y being the ſame from the nature of the queſtion) we ſhall have $2r^3 = y^3 + yr^2$; and this ſuppoſition we have certainly a right to make, ſince the numbers are taken at pleaſure, conſequently if it can poſſibly be a ſquare number, it muſt evidently hold equally good in this caſe as in any other. Now in this equation nothing can be more obvious than if $r \sqsubset y$ that $2r^3$ will be greater than $y^3 + yr^2$, and if $r \sqsupset y$ that $2r^3$ will be leſs than $y^3 + yr^2$; therefore this equation cannot poſſibly obtain in rational numbers unleſs $r = y$, and this gives $x = y$ alſo; from whence nothing more can be inferred than that the required fraction muſt be $\frac{n}{n}$, or unity; which again ſhews the abſurdity of the thing. The queſtion therefore requires an abſolute impoſſibility.

It may, perhaps, be objected by the leſs diſcerning reader, that theſe are not ſufficient proofs of the abſurdity of this queſtion, as there are vulgar fractions which are equivalent to rational ſquare numbers, and yet neither numerator nor denominator independently ſuch; and therefore $\frac{x^2-y^2}{xy}$ may be a ſquare number, when neither x^2-y^2, nor xy are rational ſquares. To which it may be anſwered, that it can only ſo happen, when the numerator and denominator of a fraction are not prime to each other, or in their loweſt terms. But $\frac{x^2-y^2}{xy}$ muſt evidently be in its loweſt terms when $\frac{y}{x}$ is ſo; it is therefore impoſſible that the expreſſion $\frac{x}{y} - \frac{y}{x}$ can be a ſquare number, unleſs both $x^2 - y^2$ and xy are ſuch, which has been before proved to be abſurd.

Another queſtion equally as abſurd, which has been ſome time handed about, is, To find a triangle ſuch, that not only the ſides may be whole numbers, but alſo a line drawn from the angle at the baſe, and terminating in the perpendicular, may be a whole number; and moreover, that the ſquare of the leſſer ſegment, taken from the ſquare of the whole perpendilar, may leave a whole ſquare number.

Now the three ſides of a triangle are only commenſurable when $m^2 + n^2$ denotes the hypothenuſe and $m^2 - n^2$, and $2mn$ the two ſides; m and n being numbers taken at pleaſure, ſo as $m \sqsubset n$. Let then AB (fig. 2.) be denoted by $2mn$, then can AD be expreſſed only by $m^2 - n^2$, if AB, AD, and BD muſt be rational numbers. And ſince $AC^2 - AD^2$ is to be a whole ſquare number, *per* queſt. AD will be alſo the ſide, and AC the hyp. of a right-angled triangle; but AD is equal $m^2 - n^2$, and therefore AC can be repreſented only by $m^2 + n^2$, to be a whole number; and ſince AB is denoted by $2mn$, AC can be equal only to $m^2 - n^2$. Hence it is evident, that if AB, AC, BC, BD, and $AC^2 - AD^2$ muſt be whole numbers, AC muſt neceſſarily be equal to both $m^2 + n^2$, and $m^2 - n^2$, which is abſurd.——But as this may not perhaps appear wholly ſatisfactory to every reader, ſince m and n may have different values, and yet $2mn$ which repreſents the baſe of the triangle continue the ſame, I ſhall conſider it a little differently,——Let AB $= 2mn$ (as before) $= 2pq$, where $m \sqsubset n$, $p \sqsubset q$, $m \sqsubset p$, and $\because q \sqsubset n$, then will BC $= m^2 + n^2$, AC $= m^2 - n^2$, BD $= p^2 + q^2$, and AD $= p^2 - q^2$; but by the queſt. $AC^2 + AD^2 = \square$,

$= \square$, that is $\overline{m^2 - n^2}|^2 + \overline{p^2 - q^2}|^2 = \square$. Now it is obvious, ſince the ſum of theſe two ſquares is to be a ſquare whole number, that their roots muſt alſo repreſent the two ſides of a commenſurate right-angled triangle; but if $p^2 - q^2$ be one ſide thereof, the other can only be equal to $2pq$ when all the ſides are rational, hence $m^2 - n^2 = 2pq = 2mn$, and AB = AC, conſequently BC ($= m^2 + n^2$) is irrational.

That AD and AC may be both expreſſed by $m^2 - n^2$, and BD and BC by $m^2 + n^2$, when the baſe is denoted by $2mn$, is evident, becauſe, as we have juſt obſerved, the factors m and n may be variouſly aſſumed, and yet the baſe of the triangle retain the ſame numerical value. From whence it appears that there may be various right-angled triangles of rational ſides, having one common baſe. To illuſtrate this, ſuppoſe we take the 18th Queſtion of the 6th Book of Diophantus's Algebra, where it is propoſed, that in the right-angled triangle ABC, AB, BC, CA, BD, AD, DC, ſhall be all whole numbers. Let $2mn$ = any compound number, ſo that m may be always greater than n; ſuppoſe 24; then will $m = 4$, or 6; and $n = 3$, or 2. From the firſt values of m and n, we have AD = 7, BD = 25, and AB = 24; and from the ſecond values of m and n, AC = 32, BC = 40, and AB = 24; and thence DC = 25.——It may here be obſerved, that this is a general method of ſolving the problem; but Diophantus is obliged to draw the line BD, ſo as to biſect the angle ABC, in order to get an equation, *viz.* AB × DC = BC × AD, to make the ſides become rational.

If we take $2mn = 28$, and $m = \sqrt{28}$, or $\sqrt{98}$, and thence $n = \sqrt{7}$, or $\sqrt{2}$, (for $2 \times \sqrt{28} \times \sqrt{7} = 28$, and $2 \times \sqrt{98} \times \sqrt{2} = 28$) we ſhall have AD = 21, BD = 35, AC = 96, BC = 100, AB = 28, and thence DC = 75; which are the numbers found by Diophantus's method.

Hence it appears, that it is equally as eaſy to find a right-angled triangle, with any number of lines drawn from either of the acute angles, and terminating in the oppoſite ſide, ſo that not only the ſides of the triangle, but alſo the lines ſo drawn, and the ſegments of the oppoſite ſide formed thereby, ſhall be all whole numbers; provided we can find a compound number to repreſent half the baſe, which may be reſolved into as many different factors.

Queſtion XXXIV.

* In triangulo plano ABD, angulus DAB obtuſus eſt, angulus DAC rectus. Dantur AD + BD = 51, AC perpendic. ad AD = 21, AB = 32. Quæritur AD. (fig. 3.)

Solut. Ex A dimitte perpendiculum AE. Sit $\mathrm{AB} = a$, $\mathrm{AC} = b$, $\mathrm{AB} + \mathrm{BD} = c$, $\mathrm{AD} = x$; tunc (ex natura triangulorum) $\frac{\mathrm{BD}^2 + \mathrm{AD}^2 - \mathrm{AB}^2}{2\mathrm{BD}}$ dat $\mathrm{DE} = \frac{c^2 - 2cx + 2x^2 - a^2}{2c - 2x}$. Et ob ſimilia triangula ACD, EAD; erit DC : DA :: DA : DE, et (*per* Euc. 47. 1.) $\mathrm{DC} = \sqrt{\mathrm{AD}^2 + \mathrm{AC}^2}$, ergo $\frac{\mathrm{AD}^2}{\overline{\mathrm{AD}^2 + \mathrm{AC}^2}|^{\frac{1}{2}}} = \mathrm{DE}$. Exinde erit $\frac{c^2 - 2cx + 2x^2 - a^2}{2c - 2x} = \frac{x^2}{\sqrt{x^2 + b^2}}$, hinc æquatio numeroſa $- 8072x^4 + 501636x^3 - 9856921x^2 + 141873228x = 1096735689$, ex hoc $x = 13{\cdot}6792$. *Q. E. D.*

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* This queſtion, with the ſolution, as they are here inſerted, were ſent to the Editor of the Town and Country Magazine in Jan. 1774, and the queſtion was accordingly propoſed in Feb. and anſwered in the next Magazine by Mr. G. B. of Coventry. But as the ſolution there given (if it may be called one) is very inelegant, and the concluſion falſe, I thought this a proper opportunity of giving the true one. It is evident, from a compariſon of the ſolutions, that he has either committed an error in the reduction of his final equation, which would ariſe to very high dimenſions; or elſe he has found the meaſure of AD from a mechanical conſtruction, which is little better than gueſſing at the anſwer.

Queſtion XXXV,

There are three ports ABE, whoſe bearings from each other are as follow, *viz.* B from A, *N* by *E* ¼ *E*; E from A, *E* by *S*, and E from B, *S. S. E.* A ſhip at the port A being bound to a certain iſland, C, bearing *N E*, ſaileth thence *E* by *S*, and after running as many miles (by the log.) as the port E is diſtant from A, in a current which ſetteth *N N E*, arrives at her deſired haven, which is 100 miles diſtant from B. Quere the diſtance of the places from each other, and the velocity of the current, without Algebra? (fig. 4.)*

Anſwer. The figure being conſtructed as *per* queſtion. In the trapezium ABCE we have BC = 100, and the ∠'s BAE, = 87°, 11′, 15″, AEB = 56°, 15′, ABE = 36°, 33′, CAE = 56°, 15′, AEC = 101°, 15′, and ACE = 22°, 30′; from whence we find BAC = 30°, 56′¼, and BEC = 45°, Now if the ſines of all the angles be drawn to any determinate radius, it will readily appear from a ſimilarity of triangles, and the compoſition of their ratios, that as the ſin. BAC × ſin. ACE : ſin. ABE × ſin. BEC :: BD : DC :: 1967 : 4202 :: 1 : 2·136,

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* This queſtion appeared in the Town and Country Magazine, for Sept. 1772, and was anſwered by the propoſer Mr. C. in the next Magazine, the above ſolution being (as I was informed) ſent too late for the month. What this Gentleman there gives us as a ſolution, may with juſt as much propriety be called ſo, as ſaying the drawing of a geometrical figure is the demonſtrating of its properties. For it plainly appears from the queſtion, that the method of inveſtigating the numerical values of the diſtances and the veloc. of the current *without having recourſe to Algebra,* was the thing required, and not a bare conſtruction of it, which, by the bye, he has alſo rendered quite falſe, by taking the ang. NAB much leſs than that given in the queſtion. The demonſtration and calculation he has omitted, becauſe, he ſays, they are too evident to be inſiſted on; but in this I cannot help thinking he is much miſtaken, for I am pretty certain that no one can ſee the reaſon of the analogy, upon which the ſolution wholly depends, without ſome recollection.

The ſum of the ang. at the baſe of the triang. BDC is 112°, 30′; therefore (*per* the third axiom of plane trigon.) it will be, as 3·136 : 1·136 :: tang. 56°, 15′, : tang. 28°, 28′, half the difference of thoſe angles; whence the ∠ CBD is 84°, 43′, and BCD 27°, 47′. Theſe being obtained, we eaſily find all the diſtances, that is, BA = 90·66, BE = 108·92, AE = 6[illegible]·94, AC = 166·11, and EC = 140·82. And ſince EC is ∥ to A*a* (the direction of the current) *per* conſtruc. therefore as AE : EC :: 1 : 2·13, which is the ratio of the ſhip's velocity to the velocity of the current, which are all obtained without Algebra. *W. W. R.*

Queſtion XXXVI.

"In a plane triangle, having given the vertical angle, the difference of the baſe and one ſide, and the ſum of the perpendicular, from the angle of the baſe contiguous to that ſide upon the oppoſite ſide, and the ſegment thereby cut off from that oppoſite ſide contiguous to the other angle at the baſe; to conſtruct the triangle*."

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* This queſtion was propoſed in the Ladies' Diary for 1774, by the Rev. Mr. Lawſon, to which two different ſolutions were inſerted in the laſt year's Diary by the Rev. Mr. Wildbore, and the author of this Treatiſe. But as neither of theſe ſolutions were conciſe enough for the conductor of the mathematical correſpondence in the Town and Country Magazine, he repropoſed it; hoping that ſome of his ingenious contributors would ſend a more elegant ſolution. How far this has been performed I ſhall not pretend to determine; but the method of conſtruction given in the Town and Country Magazine is certainly more ſimple than either of thoſe in the Diary. But as the figure itſelf is quite falſe, owing to a want of paralleliſm in the lines, the triangle produced does not in any one reſpect correſpond to the data; the vertical angle being much too large, and the aſſumed difference in the queſtion being double of that in the triangle. Upon this account, and the want of a demonſtration, which perhaps may not appear ſo evident (at leaſt to the young reader) as he ſeems to think, I have inſerted it here and corrected the figure.

CONSTRUCTION.

" Make AB (fig. A) equal to the ſum of the perpendicular and ſegment, the angle ABC, 45°, and ABE the ſupplement of the given vertical one, BE the given difference. Produce CB to meet FED drawn parallel to AB, in D. Join the points A, D, draw BH = BE; from A draw AC and CG parallel to BH and BE reſpectively, then will AGC be the triangle required."

DEMONSTRATION.

Draw BK $\parallel$ to DA; and join EH, and GK. Then, ſince AB is equal to the given ſum of the perpendicular and ſegment, and the $\angle$ ABC half a right angle, by conſtruction, a perpendicular let fall from any point as C upon the oppoſite ſide will be equal to aB (Eu. 9. 4.) $\therefore$ Aa + aC = AB, the given ſum. And as the $\angle$ ABE is equal to the ſup. of the given $\angle$ by conſtruc. and CG $\parallel$ to BE, the $\angle$ CGB is equal to the $\angle$ ABE (Eu. 29. 1.) and therefore the $\angle$ CGA equal to the given vertical angle. And moreover, becauſe DE, DH, DB, BH, BE, are reſpectively $\parallel$ to BG, BK, BC, CK, CG, the triangles BEH, CGK are ſimilar. But BH is = BE $\therefore$ CK is = CG. And again, becauſe DA is $\parallel$ to BK, and KA $\parallel$ to BH, KA is = BH = BE = the given difference of the baſe CA, and one ſide CG. *Q. E. D.*

Queſtion XXXVII.

In the mound of an Elliptical Garden, whoſe tranſverſe is to the conj. as 3 to 2, a pedeſtal, 50 yards high, is ſo placed, as that the apparent magnitude of an Herculean Figure, 10 feet high, on the top of it, is the greateſt poſſible to an eye ſituated on the tranſverſe, and 20 yards from the center of the Ellipſe. Required from hence the Area of the Garden.

Anſwer. In Fig. 5. EF repreſents the pedeſtal $\perp$ to the plane of the ſemi-ellipſe ABI; FG the ſtatue, and D the given point in the tranſverſe. Now it is evident ſince DE is in

Fig. 1

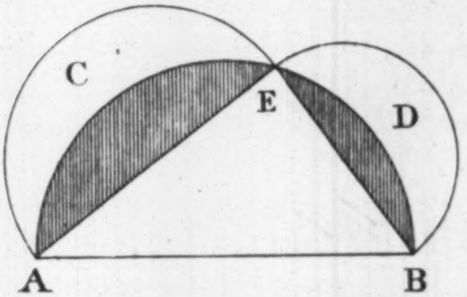


Fig. 2

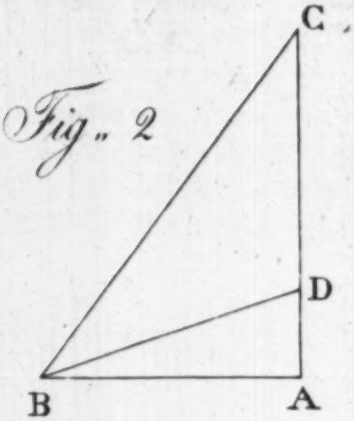


Fig. 3

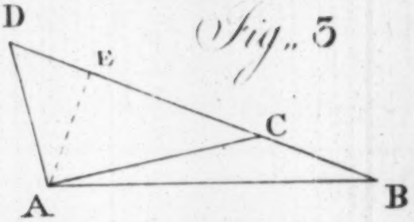


Fig. 4

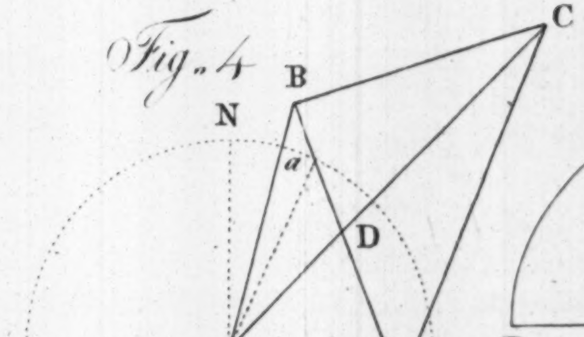
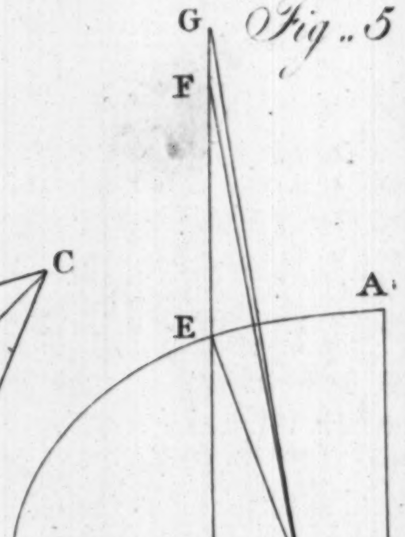


Fig. 5



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the plane of the ellipſe, and GE ⊥ to that plane, that the ∠ GED is a right one, and therefore if we call GE rad. ED will be the tang. of the ∠ G, and $\frac{ED \times GE}{FE}$ that of the ∠ F. Moreover it is evident, that the ∠ FDG under which the object is ſeen, is the difference of the angles DFE, DGE, which muſt be a max. *per* queſtion. Making therefore FE $= a$, EG $= b$, and ED $= t$, we have $\frac{bt}{a} =$ the tang. of the greater angle, and $t =$ that of the leſs; and the flux. of the difference of theſe angles, expreſſed in terms of the rad. and tang. is $\frac{b^2\dot{t}}{b^2 + t^2} - \frac{ab^3\dot{t}}{ab^2 + b^2t^2}$, which being made $= 0$, and reduced, gives $t = \surd ab$. Hence it appears that DE the diſt. of the pedeſtal is a mean proportional between GE and FE.

Now, it is obvious, from the nature of the queſt. that the line DE muſt be the neareſt diſt. to the curve, or a min. therefore, putting ID $= c$, and the ſem. tranſv. $= x$, we have (*per* queſt. and *Emerſon's* Flux. *p.* 126) $\frac{4c}{5} =$ DH; hence (Euc. 47. 1.) $ab - \frac{16c^2}{25} = \text{HE}^2$, and *per* prop. of the Ellipſe, $\frac{4}{9} \times \overline{x^2 - \frac{81c^2}{25}} = \text{HE}^2$; $\because \frac{4}{9} \times \overline{x^2 - \frac{81c^2}{25}} = ab - \frac{16c^2}{25}$; reduced $x = 245 \cdot 9385$, and thence the area of the Ellipſe $= 126681 \cdot 1$ ſquare feet *.

Queſtion

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* This queſtion I propoſed in the Ladies' Diary for 1775, to which a ſolution is given in the Diary for the year following by Mr. Rowe; but, by a miſtake in the concluſion of it, he has brought out the required area of the ellipſe only about one half what it ought to be.

The ſame queſtion was re-propoſed by the conductor of the mathematical department in the T. and C. Mag. for Aug. 1776, occaſioned chiefly, I imagine, by the miſtake in Mr. Rowe's ſolution. —— And here, in order to do juſtice to the author of the Diary and myſelf, I am under the diſagreeable neceſſity of informing the reader, that the note ſubjoined to the

ſolution

Queſtion XXXVIII.

From the fluxionary expreſſion $\sqrt{16y^4.\overline{a^2-y^2}.\dot{x}^4\dot{y}^4} - 4d\dot{x}\dot{y}^3 = 0$. Required a finite value of x in terms of y and known quantities.

Anſwer. The expreſſion by tranſpoſition, &c. becomes $4\dot{x}^2\dot{y}^2y^2\sqrt{a^2-y^2} = 4dx\dot{y}^3$. Divide by $4\dot{x}\dot{y}^2$, and there ariſes $\dot{x}y^2\sqrt{a^2-y^2} = d\dot{y}$; hence $\dot{x} = \dfrac{d\dot{y}}{y^2\sqrt{a^2-y^2}}$. Now by

ſolution of this queſtion in the Mag. for the month following, contains aſſertions that I never wrote or even once thought of; which the following letter will evince, being copied verbatim from that I ſent with the above ſolution to the Editor of the T. and C. Magazine.

" Sir,

" As it will be naturally expected that I ſhould offer a ſolution to my
" own queſtion, I have here ſent one nearly the ſame with that I gave to
" the author of the Diary.——The reſtriction you ſpeak of is certainly
" right, but I thought it rather unneceſſary to mention it, becauſe it is
" obvious to every one that either this limitation muſt be underſtood, or
" the ſituation of the pedeſtal in the periphery given; otherwiſe the queſtion
" would be abſolutely unanſwerable.——Mr. Rowe's ſolution is on the
" ſame principle as the above; but is rendered falſe, I obſerve, by a miſ-
" take in the concluſion of it, having put down $3 \times \cdot 7854 \times \overline{ab + \frac{4}{3}cc}$
" inſtead of $6 \times \cdot 7854 \times \overline{ab + \frac{4}{3}cc}$, this latter expreſſion bringing out the
" ſame numbers as the above.

" I am, &c."

I remember Terence ſomewhere ſays, *Veritas odium parit.*—which perhaps may be the caſe here; but notwithſtanding this, I cannot help obſerving, that whoever compares this letter with the note before-mentioned will hardly forbear tearing it out of the Mag. as a downright falſehood. ——It is a pity that this entertaining and curious miſcellany ſhould be proſtituted to ſuch low purpoſes in the Mathematical parts, as to be a vehicle for malevolence and ſpleen; the repropoſing of this queſt. in the Mag. being evidently only another weak effort to depreciate (if poſſible) the mathematical character of the preſent author of the Ladies' Diary; for which purpoſe, it ſeems, truth or falſehood is equally ſubſervient.

Emerſon's Tables, Forms 10th and 12th, we get the fluent of $\frac{\dot{y}}{y^2\sqrt{a^2-y^2}} = -\frac{\sqrt{a^2-y^2}}{aay}$; and therefore the fluent of $\frac{d\dot{y}}{y^2\sqrt{a^2-y^2}} = \frac{-d\sqrt{a^2-y^2}}{a^2y}$, whence $x = -\frac{d\sqrt{a^2-y^2}}{a^2y}$; the Neg. ſign ſhewing that it would require a correction according to the nature of the problem.

Queſtion XXXIX.

In the expreſſion $\frac{zy\dot{y}^3}{\dot{z}^2}$, required the relation of y and z, ſuppoſing their naſcent increments to be cotemporaneous, and the fluent correſponding to any given values thereof a minimum; alſo the curve defined by the equation expreſſing the relation of y and z.

Anſwer. Let $\dot{y}$ alone be conſidered as variable, then the Flux. will be $\frac{3zy\ddot{y}\dot{y}^2}{\dot{z}^2}$; and when y only is made variable, its Flux. becomes $\frac{z\dot{y}^4}{\dot{z}^2}$. Let now the latter be divided by the former, and the quotient put $= \frac{\dot{t}}{t}$, and we have $\frac{\frac{1}{3}\dot{y}}{y} = \frac{\dot{t}}{t}$: Hence the Hyp. Log. t = Hyp. Log. $\sqrt[3]{y}$ + Hyp. Log. m, (m being any conſtant quantity) therefore $t = my^{\frac{1}{3}}$; which being equated with the Flux. of the given expreſſion, when $\dot{y}$ alone is variable, gives $\frac{3zy\ddot{y}\dot{y}^2}{\dot{z}^2} = my^{\frac{1}{3}}$, from whence $y^{\frac{2}{3}}\dot{y}^2 = \frac{1}{3}mz^{-1}\dot{z}^2$, the Fluent of which is $\frac{3}{4}y^{\frac{4}{3}} = 2\sqrt{\frac{1}{3}mz}$, expreſſing the relation of y and z. If for $\left.\frac{8}{3}\sqrt{\frac{m}{3}}\right|^{\frac{6}{5}}$ we put d, then the equation becomes $y^8 = d^5z^3$ anſwering to a Parabola of the higher order.

Queſtion XL.

Required the lat. of the place, and declination of the ſun, when the length of the day is to that of the night, in the ratio of 3 to 2, and the ſun's meridian altitude to his depreſſion at midnight, as 2 to 1. See T. and C. Mag. *p.* 303. 1773.

Anſwer. In the Orthographic ſcheme (fig. 6.) P repreſents the North Pole, EQ the Equator, HO the Horizon, AD the Sun's Semidiurnal Arc, and DR the Seminocturnal. From the given ratio of the lengths of the day and night, the arches AD, DR, are both known; therefore putting a = the nat. verſed ſine of AD, b = the nat. v. ſ. of DR, and x = (Rb) the nat. ſine of ⊙ R the Sun's depreſſion at midnight, *per* ſim. △'s $b : a :: x : \frac{ax}{b}$ = Aa, the ſine of the Sun's merid. altitude. But *per* queſt. AH = 2 OR, that is $2x\sqrt{1-x^2}$ = Aa, ∴ $\frac{ax}{b} = 2x\sqrt{1-x^2}$; hence $x = \sqrt{1-\frac{a^2}{4b^2}} = \cdot320604$ the nat. ſine of 18°, 42′, the Sun's depreſſion at midnight; ∴ by the nat. of the ſphere the comp. of the req. lat. HE $(= \frac{1}{2} \times \overline{\text{HA} + \text{OR}}) = 28°, 3'$, and the Sun's declin. = EA $(= \frac{1}{2} \times \overline{\text{HA} - \text{OR}}) = 9°, 21'$. *W. W. R.*

Queſtion XLI.

The Fraction $\frac{2\acute{1}3\cdot47\acute{6}}{4\acute{1}7\cdot\acute{6}}$ is equal to $511176\,\frac{314\frac{26586}{100000}}{417\cdot2}$. Required the inveſtigation.

Queſtion XLII.

If any plane triangle ABC (fig. 7.) be circumſcribed by a circle, and a right line be drawn from any one of the angular points, ſuppoſe B, biſecting the ſaid ang. till it meets the circumference in D: I ſay that a circle deſcibed with rad.

DA, will paſs through the center of a circle inſcribed in that triangle, and alſo through the other angular point C. Required a Geometrical demonſtration.

Queſtion XLIII.

To find the center of a circle P (fig. 8.) to paſs through a given point, and to cut two lines given in poſition, ſo that the intercepted arch AB may be of a given magnitude; or, that its chord may ſubtend a given ∠ APB.

Queſtion XLIV.

From a given point C (fig. 9.) on the diam. AB of a circle produced, let a tang. CD be drawn, and another right line from the point C, cutting the periphery in E and F, ſo that the ſine of the ∠ BCE, may be a fourth proportional to CB, (conſidered as radius) and the ſines of BCD and a given ∠ P. I ſay that the acute ∠ AIF or BIE, formed by the diagonals AE, BF, is equal to the given ∠ P. Required a Geometrical demonſtration.

Queſtion XLV.

Suppoſe a ſemicircular bowl placed on an horizontal plane, at what height and diſtance from the bowl muſt a light be placed, ſo as to illuminate one half the interior ſurface thereof?

Queſtion XLVI.

By repeated obſervations on the Northernmoſt ſtar in the right foot of Urſa Major (marked by Bayer ι) whoſe preſent declination is 49°, 15′, I have found that its altitude encreaſes more in a given time, in a certain latitude, than any other ſtar of different declination. Required the latitude of the place of obſervation, and the increaſe of the altitude of the ſaid ſtar ι from 8 to 11 when it ſouths at midnight.

Queſtion XLVII.

In the latitude 53°, 27′, N. on June 25, 1775, I ſaw a rainbow bearing *ESE*. Required the hour of the day.

Queſtion XLVIII.

Suppoſe an inflexible rod of iron, 40 feet long and 2 inches diameter, be ſo fixed at one end that it may vibrate freely: In what time will the other end of the rod, being let fall from an horizontal direction, deſcribe an arch of a given length, ſuppoſe 35 feet, from the commencement of motion?

Queſtion XLIX.

Required general expreſſions for the ſides of a right-angled triangle in whole numbers, ſo that any given number of lines drawn from one of the acute angles and terminating in the oppoſite ſide, as alſo the ſegments of the ſaid ſide formed thereby, may be all whole numbers. And moreover it is required, that the given number of lines ſo drawn may be the moſt which that triangle can poſſibly admit of in whole numbers; and that the periphery of the triangle may be a minimum. Suppoſe 5 the number of lines to be drawn, required the ſides of the triangle, &c. as above.

Queſtion L.

A cone being cut by a plane parallel to the baſe, the area of the ſection (= 20·7736 inches) is found to be a mean proportional between the ſuperficies of the two parts of the cone; and the ratio of the ſide of the lower fruſtum to the ſemidiameter of the baſe as 23 : 80. Required the dimenſions of the cone.

Queſtion

Fig. 6

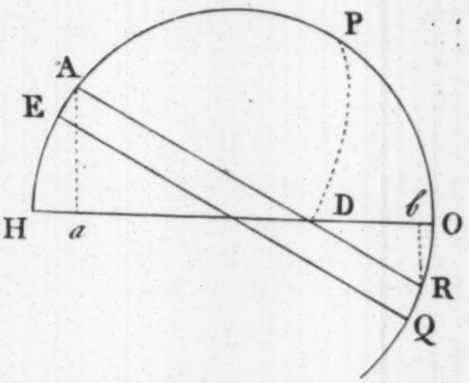


Fig. 7

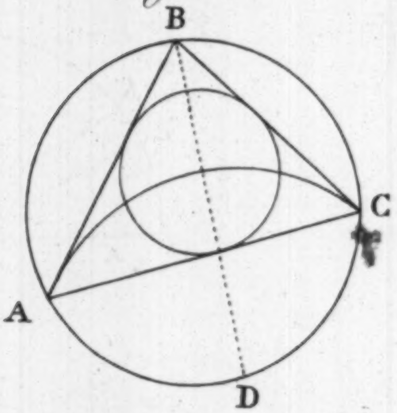


Fig. 8

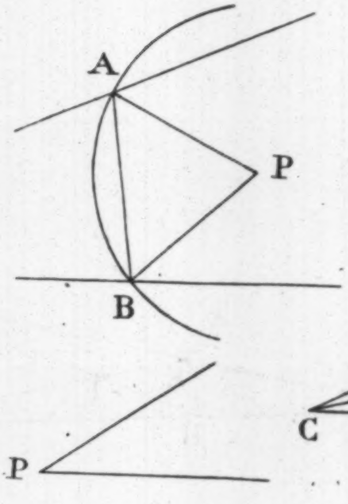


Fig. A

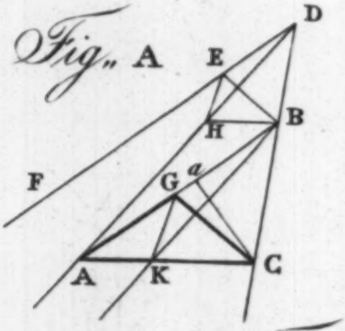
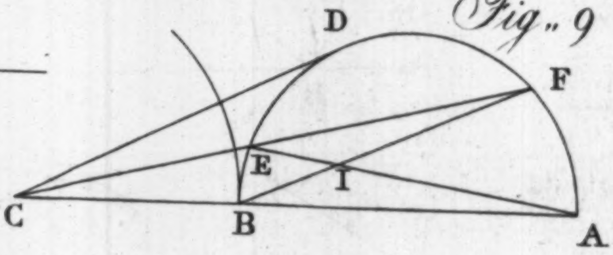


Fig. 9



Queſtion LI.

The expreſſion $\sqrt{ax^3 - b}$, is a rational number. Required a rational value of x.

Queſtion LII.

The internal diameter and diagonal of a cylindrical caſk (which is made of the leaſt wood poſſible) are reſpectively expreſſed by x^y, and $\overline{\overline{x^y}\big|^x \times x}\Big|^{\frac{1}{2}}$. Required the content in Ale Gallons when $y = x^x$.

Queſtion LIII.

Let $y - a = \overline{a^2 \times \overline{a - x^2}}\Big|^{\frac{1}{4}}$ be an equation to a curve. Quere the Abſciſſa when y is a maximum.

Queſtion LIV.

How many elementary ſounds may be formed out of the 24 letters of the alphabet[*] ?

Queſtion LV.

It is required to divide the area of a circle geometrically into a given number of parts, which may be equal both in area and circumference[†].

Some uſeful Remarks upon Equations.

As the young Algebraiſt generally meets with ſome difficulty in rightly aſcertaining the roots of quadratic equations of the

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[*] Tacquet in his *Arithmeticæ Theor.* p. 381, ſays. Mille milliones ſcriptorum mille annorum millionibus non ſcribent omnes 24 litterarum alphabeti permutationes, licet ſinguli quotidiè abſolverent 40 paginas, quarum unaquæque contineret diverſos ordines litterarum 24.

[†] This *Paradoxical Queſtion*, or perhaps rather *double entendre*, is taken from *Lawſon's Diſſertation on the Geometrical Analyſis of the Antients.*—— The demonſtrations of all the theorems, with the Geometrical Conſtructions of the problems contained in this book, will be given at the latter end of *An Eſſay on the uſefulneſs of Mathematical Learning*, which will ſoon be publiſhed.

3^d form, as alſo in Cardan's and Colſon's Formulæ for the roots of Cubics and Biquadratics, from the irrational Binomial $\pm x \pm \sqrt{-y}$. I ſuppoſe the following remarks will not be unacceptable to him.

1. In the ſolution of a problem where the final equation is in this form $x^2 - ax = -b$, and thence $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$, the conditions of the problem muſt determine whether the affirmative or negative ſign gives the true value of x; for if from the nature of the queſtion, x be greater than a, we muſt evidently uſe the affirmative ſign; and if leſs, the negative. As in this problem. *To divide the number* 100 (a) *into two ſuch parts that their product and the difference of their ſquares may be equal to each other.* If we denote the leſſer part by x, and therefore the greater by $a - x$, we ſhall find $x = \frac{3a}{2} \pm \sqrt{\frac{5aa}{4}}$. But x being given leſs than a, the upper ſign (+) gives x too great; ſo that $x = \frac{3a}{2} - \sqrt{\frac{5aa}{4}} = 38{\cdot}19658$, &c. muſt be the true value required.

2. Hence it appears that though there be two affirmative roots in a quad. equation, yet, in general, only one of them will anſwer one caſe, or the particular queſtion propoſed. The ſame obſervation holds good in equations of all dimenſions; for ſuppoſe in the ſolution of a queſtion we have derived this final biquadratic equation, $x^4 - ax^3 + bx^2 - cx + d = 0$, where all the roots are affirmative, we muſt not conclude that the queſtion admits of four different anſwers, for it will often be found upon trial, that three of the roots will produce an abſurdity, and only one value anſwer the particular caſe propoſed.

3. In a queſtion producing a quad. equation of the third form, if the unknown quantity be aſſumed indeterminately in regard to *greater* and *leſs*, then will the affirmative and negative ſigns exhibit thoſe values reſpectively. For if it was required to find two numbers whoſe ſum is a and the ſum of their alternate quotients b, we ſhall find that $\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \frac{a^2}{2+b}}$ are the numbers required.

4. In the reduction of cubic equations it will be proper to inform the young reader, that Cardan's rule is only of uſe in caſes where two of the three roots are impoſſible; and therefore it would be in vain to attempt to ſolve a cubic equation compoſed of real roots by this method. As for example. Let the equation $z^3 - 7z = 6$ be propoſed, whoſe roots are -2, 3, and -1; the numeral coefficients being written in the formula, we have $z = \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{72}}}$; which is only an imaginary expreſſion; the ſquare root of a negative quantity being impoſſible. So in this equation $x^3 - 91x = -330$ the roots of which are 5, 6, and -11, we get $x = \sqrt[3]{-165 + \sqrt{-685{\cdot}03}} + \sqrt[3]{-165 - \sqrt{-685{\cdot}03}}$, which is alſo imaginary. But when two of the roots are impoſſible as in this equation $x^3 + 6x = 20$; then we get $x = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = 2$, where the expreſſion is real and poſſible; the other values of x being imaginary.

5. I ſhall now endeavour to clear up to the learner ſome ſeeming difficulties in finding the roots of a cubic equation by Colſon's Theorem. And as this method principally depends

upon finding the cubic root of an impoſſible binomial, I ſhall firſt ſhew the inveſtigation of an eaſy rule for obtaining ſuch roots.

Let $a + \sqrt{b} = \sqrt[3]{A + B}$; then by involution $A + B = a^3 + 3a^2\sqrt{b} + 3ab + b^{\frac{3}{2}}$. Put $a^3 + 3ab = A$, and $3a^2\sqrt{b} + b^{\frac{3}{2}} = B$; then will $\overline{a^3 + 3ab}|^2 - \overline{3a^2\sqrt{b} + b^{\frac{3}{2}}}|^2 = a^6 - 3a^4b + 3a^2b^2 - b^3 = \overline{a^2 - b}|^3 = A^2 - B^2$, $\therefore b = aa - \overline{A^2 - B^2}|^{\frac{1}{3}}$; hence, by ſubſtitution, $a^3 + 3a \times \overline{a^2 - \overline{A^2 - B^2}|^{\frac{1}{3}}} = A$, that is $4a^3 - 3\sqrt[3]{A^2 - B^2} \times a = A$; from whence a is eaſily found, and ſince $\sqrt{b} = \sqrt{a^2 - \sqrt[3]{A^2 - B^2}}$, we have $a + \sqrt{a^2 - \sqrt[3]{A^2 - B^2}} = \sqrt[3]{A + B}$.

When $\sqrt[3]{A^2 - B^2}$ is a ſurd, both members of the root will be irrational. In that caſe multiply the given equation by ſome number till $\sqrt[3]{A^2 - B^2}$ comes out rational, remembering to divide the values of a and $\sqrt{b}$ at the laſt by the root of that number.

6. Now if the ſecond term of any cubic equation, reduced to Colſon's general form, be exterminated, we ſhall have by Cardan's rule this value of the new root,

$$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}.$$

And as the cubic root of a binomial may be always inveſtigated in a ſimilar form, they aſſume $m \pm \sqrt{n} = \sqrt[3]{r \pm \sqrt{r^2 - q^3}}$, and thereby obtain $z = m + \sqrt{n} + m - \sqrt{n} = 2m$; but as this gives but one value of z, they derive two more expreſſions from this moſt obvious principle, *viz.* that the cube root of any number, being multiplied by the cube root of unity, muſt ſtill remain a root of that number, that is, $\sqrt[3]{x^3} \times \sqrt[3]{1} = x$; but the three roots of unity are 1, $\frac{-1+\sqrt{-3}}{2}$, and $\frac{-1-\sqrt{-3}}{2}$: ſo it appears that the cube root of each binomial may be expreſſed in three different forms as follow:

$$\sqrt[3]{r+\sqrt{r^2-q^3}} =$$

$$\text{I. } \overline{m+\sqrt{n}} \times 1 = m+\sqrt{n}.$$

$$\text{II. } \overline{m+\sqrt{n}} \times \frac{-1+\sqrt{-3}}{2} = \frac{-m-\sqrt{n}+m\sqrt{-3}+\sqrt{-3n}}{2}.$$

$$\text{III. } \overline{m+\sqrt{n}} \times \frac{-1-\sqrt{-3}}{2} = \frac{-m-\sqrt{n}-m\sqrt{-3}-\sqrt{-3n}}{2}.$$

$$\sqrt[3]{r-\sqrt{r^2-q^3}} =$$

$$1.\ \overline{m-\sqrt{n}} \times 1 = m-\sqrt{n}.$$

$$2.\ \overline{m-\sqrt{n}} \times \frac{-1+\sqrt{-3}}{2} = \frac{-m+\sqrt{n}+m\sqrt{-3}-\sqrt{-3n}}{2}.$$

$$3.\ \overline{m-\sqrt{n}} \times \frac{-1-\sqrt{-3}}{2} = \frac{-m+\sqrt{n}-m\sqrt{-3}+\sqrt{-3n}}{2}.$$

From whence we ſhall evidently have theſe nine different expreſſions for z,

$$\text{I.} + 1. = 2m \text{ - - - - - - - - - - } = z.$$

$$\text{I.} + 2. = \frac{m+3\sqrt{n}+m\sqrt{-3}-\sqrt{-3n}}{2} \text{ - . - } = z.$$

$$\text{I.} + 3. = \frac{m+3\sqrt{n}-m\sqrt{-3}+\sqrt{-3n}}{2} \text{ - - } = z.$$

$$\text{II.} + 1. = \frac{m-3\sqrt{n}+m\sqrt{-3}+\sqrt{-3n}}{2} \text{ - - } = z.$$

$$\text{II.} + 2. = m+m\sqrt{-3} \text{ - - - - - - } = z.$$

$$\text{II.} + 3. = -m+\sqrt{-3n} \text{ - - - - - - } = z.$$

III. + 1. = $\frac{m - 3\sqrt{n} - m\sqrt{-3} - \sqrt{-3n}}{2}$ - = z.
III. + 2. = $-m - \sqrt{-3n}$ - - - - - = z.
III. + 3. = $-m - m\sqrt{-3}$ - - - - - = z.

But as the 6th and 8th expreſſions only correſpond in the form to the cubic root of the irrational binomial, they are therefore the moſt convenient for uſe.

7. Having compared the given equation with the general one and thence obtained the numerical values of q and r, it will then appear, whether $r^2 - q^3$ be affirmative or negative. And ſince it is evident from the rule, that whatever ſign the value of $r^2 - q^3$ is affected with in the binomial, the ſame ſign will the correſponding term (n) have in the root, it plainly follows, that if n (or $r^2 - q^3$) be negative the expreſſions $-m + \sqrt{-3n}$, $-m - \sqrt{-3n}$ will become real and poſſible, that is $-m + \sqrt{3n}$, $-m - \sqrt{3n}$, which no one can be ſo dull as not to ſee the reaſon of, if he knows that a negative quantity ſubtracted becomes an affirmative one. On the contrary if n be affirmative, $\sqrt{-3n}$ is abſolutely impoſſible, there being no ſuch thing (according to the common definition of the term) as the ſquare root of a negative quantity; thoſe two roots are therefore then imaginary, and $2m$ is the only poſſible value of z.

8. It appears from the laſt Rem. that Cardan's Theorem may be rendered generally uſeful, by ſolving ſuch equations as have three roots real, as well as thoſe that have but one root real and two impoſſible ones. For ſince the three values of z may be expreſſed by $2m$, $-m + \sqrt{-3n}$, and $-m - \sqrt{-3n}$, it follows, that whenever the expreſſion for the root becomes imaginary, by ſubſtituting the numerical values in the formula, that is when $r^2 - q^3$ is negative, it

will be poſſitive in the formulæ for the cubic roots; and therefore Cardan's may be as conveniently uſed as Colſon's form.

Example 1.

Let $x^3 - 12x^2 + 41x = 42$ be propoſed. The ſecond term being exterminated, we have $z^3 - 7z = 6$; this, by ſubſtitution in the formula, becomes

$$z = \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}.$$

Here then by Rem. 5. we have $A = 3$, $B = \sqrt{-\frac{100}{27}}$, and $\sqrt[3]{A^2 - B^2} = \sqrt[3]{\frac{343}{27}} = \frac{7}{3}$, $\because 4a^3 - 7a = 3$; hence $a = -1 = m$, and $\sqrt{a^2 - \sqrt[3]{A^2 - B^2}} = \sqrt{1 - \frac{7}{3}} = \sqrt{-\frac{4}{3}} = \sqrt{n}$; theſe values being written in the formulæ for z give

$$z = 2m \text{ - - - - - - - - - - - } = -2.$$
$$z = -m + \sqrt{-3n} = +1 + \sqrt{4} = +3.$$
$$z = -m - \sqrt{-3n} = +1 - \sqrt{4} = -1.$$

From whence the roots of the propoſed equation are,

$$x = -2 + 4 = 2.$$
$$x = +3 + 4 = 7.$$
$$x = -1 + 4 = 3.$$

Example 2.

Let the propoſed equation be $x^3 - 21x = -20$.

Here $x = \sqrt[3]{-10 + \sqrt{-243}} + \sqrt[3]{-10 - \sqrt{-243}}$.

rom whence $\sqrt[3]{A^2 - B^2} = 7$; therefore $4a^3 - 21a = -10$, nd the root $a = 2$, and $\sqrt{n} = \sqrt{-3}$; hence the three oots are,

$$x = 2m \text{ - - - - - - - - - - - } = +4,$$
$$x = -m + \sqrt{-3n} = -2 + \sqrt{9} = +1,$$
$$x = -m - \sqrt{-3n} = -2 - \sqrt{9} = -5.$$

Example 3.

Given $x^3 - 15x = 4$, required x.

The numbers being ſubſtituted in the formula, and the cubic roots extracted as before, we get, $2 + \sqrt{-1} = m + \sqrt{n}$; from whence the roots are found to be,

$$x = 2m = \text{ - - - - - - - - - - - } 4,$$
$$x = -m + \sqrt{-3n} = \text{ - - } -2 + \sqrt{3},$$
$$x = -m - \sqrt{-3n} = \text{ - - } -2 - \sqrt{3}.$$

9. One root of any cubic equation may be had by Cardan's Form though the expreſſion becomes impoſſible, without having recourſe to the formulæ $m \pm \sqrt{n}$, by extracting the roots of the binomials, the impoſſible terms vaniſhing in the addition by being always affected with contrary ſigns.

As for example in this equation, $y^3 - 7y = 6$; here $y = \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}$ which is an impoſſible expreſſion; but the cubic roots being extracted we get $-1 + \sqrt{-\frac{4}{3}} - 1 - \sqrt{-\frac{4}{3}} = -2$, a root.

So in this equation $y^3 - 15y = 4$, we have $y = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$; which is alſo imaginary; but the cubic roots being extracted, we ge $2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$, a root.

Suppoſe it is required to exhibit, by Cardan's Rule, the thre roots of this equation $x^3 + \sqrt[3]{54} \times x = 44$, where two o the roots are impoſſible and $\sqrt[3]{A^2 - B^2}$ irrational.

B

By the rule we find $x = \sqrt[3]{22 + \sqrt{486}} + \sqrt[3]{22 - \sqrt{486}}$; and by Rem. 5. $\sqrt[3]{A^2 - B^2} = -\sqrt[3]{2}$, a ſurd. Multiply by 2, and the value of x becomes $\sqrt[3]{\frac{44 + \sqrt{1944}}{2}} + \sqrt[3]{\frac{44 - \sqrt{1944}}{2}}$; hence $\sqrt[3]{A^2 - B^2} = -\sqrt[3]{8} = -2$, $\because 4a^3 + 6a = 44$, and the root is $a = 2$; ſo $\sqrt{b} = \sqrt{6}$, and $2 + \sqrt{6} + 2 - \sqrt{6} = 4$. Therefore, $\frac{4}{\sqrt[3]{2}} = 2^{\frac{5}{3}}$ is the real root of the propoſed equation, and the two impoſſible roots are $\frac{-2 + \sqrt{-18}}{\sqrt[3]{2}}$, and $\frac{-2 - \sqrt{-18}}{\sqrt[3]{2}}$. For $\overline{x + \frac{2 - \sqrt{-18}}{\sqrt[3]{2}}} \times \overline{x + \frac{2 + \sqrt{-18}}{\sqrt[3]{2}}} \times \overline{x - 2^{\frac{5}{3}}} = x^3 + \sqrt[3]{54} \times x = 44$, the propoſed equation.

10. As the ſolutions of biquadratics depend upon the cubic or quad. equations, there can no more difficulties occur therein than thoſe we have already explained. In regard to general formulæ for the ſurſolid and higher equations, I think they may, as yet, remain among the Mathematicians *deſiderata*, as there does not appear to be any method of obtaining ſuch finite expreſſions, without ſome particular relation of the roots, or of ſome of the coefficients. I ſhall give an example or two of this kind, where there is a particular relation of the coefficients to the abſolute number; or of one coefficient to the reſt.

Example 1.

Let $x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$, and let $E = \frac{DA}{2} - \frac{CA^2}{4} + \frac{BA^3}{8} - \frac{A^5}{32}$; then will the roots of the following equations be thoſe of the propoſed one, *viz.*

$$x^4 + px^3 + qx^2 + rx + s = 0, \text{ and } x + p = 0.$$

Where $p = \frac{A}{2}$, $q = B - \frac{A^2}{4}$, $r = C - \frac{AB}{2} + \frac{A^3}{8}$, and $s = D - \frac{CA}{2} + \frac{A^2B}{4} - \frac{A^4}{16}$. For the two equations multiplied produce

$$\left.\begin{matrix} x^5 + p \\ + p \end{matrix}\right\} \times x^4 \left.\begin{matrix} + q \\ + pp \end{matrix}\right\} \times x^3 \left.\begin{matrix} + r \\ + pq \end{matrix}\right\} \times x^2 \left.\begin{matrix} + s \\ + pr \end{matrix}\right\} \times x + ps = 0.$$

And the coefficients being equated, we get the values of p, q, r, s, and the particular relation of E as above.

Let this formula be applied to the numeral ſurſolid equation $x^5 + 26x^4 + 225x^3 + 820x^2 + 1244x + E = 0$, where the coefficients are aſſumed at pleaſure. Here then $A = 26$, $B = 225$, $C = 820$, $D = 1244$, and therefore $E \left(= \frac{DA}{2} - \frac{CA^2}{4} + \frac{BA^3}{8} - \frac{A^5}{32}\right) = 624$, the abſolute number; hence $p = 13$, $q = 56$, $r = 92$, and $s = 48$; then will the roots of the biquadratic $x^4 + 13x^3 + 56x^2 + 92x + 48$, be found to be -1, -2, -4, and -6, and conſequently thoſe of the propoſed equation will be -1, -2, -4, -6, and -13.

Example 2.

Suppoſe the ſecond term exterminated, and let the reſulting equation be $z^5 + Bz^3 \pm Cz^2 \pm Dz + E = 0$, and let $D = BR^2 - \frac{RE}{B}$, R being the root of this equation $a^3 - Ba = \frac{E}{B} - C$; then will the five values of z be expreſſed by the roots of theſe two equations $z^3 + az^2 + a^2z + \frac{c}{d} = 0$, $z^2 - az + d = 0$. For theſe equations being multiplied produce an equation ſimilar to that propoſed, and the value of a is known from the above cubic equation; and by equating the coefficients we have alſo the values of b and c.

When this formula is applied to any numerical ſurſolid equation; as $z^5 + 16z^3 - 820z^2 + Dz + 320 = 0$, (the coefficients being aſſumed at pleaſure) we firſt find the value of $a = 10$, and thence we get $D = 1440$, $d = 16$, and $\frac{c}{d} = 20$. Therefore the roots of this equation $z^5 + 16z^3 - 820z^2 + 1440z + 320$, will be had by reſolving the cubic, $z^3 + 10z^2 + 100z + 20 = 0$, and the quadratic $z^2 - 10z + 16 = 0$, the roots of which are, $+ 9{\cdot}93$, $+ 8$, and $+ 2$, the other two roots being imaginary.

11. Various other literal expreſſions may be found, by taking away ſome intermediate term, and then finding two equations, the product of which ſhall give a reſult ſimilar to the propoſed equation; but theſe will be found to be of uſe only in particular caſes, which ſeldom occur in practice. For though there may be always as many independent equations as unknown quantities, yet it will be found, that by their different combinations, we ſhall always recur to an equation of the ſame dimenſions with that of which we are endeavouring to inveſtigate the roots. And if we ſtrike out any one term in either of the aſſumed generating equations, we ſhall then have more equations than unknown quantities, from whence neceſſarily ariſes a particular relation between ſome of the coefficients.

12. But ſuppoſing a compleat formula for the roots of a ſurſolid, or an equation of higher dimenſions, were by any algebraic artifice obtained, it would be of little value, for there would be far more trouble in getting the roots this way,

than by the method of converging ſeries, which is well known to be generally far more expeditious, even in biquadratics and cubics, than the finite theorems. The following general formula for ſurſolid equations I have deduced from thence, wherein the method of operation is rendered ſo ſimple, that I think none more eaſy need be wiſhed for.

If $x^5 + ax^4 + bx^3 + cx^2 + dx + m = 0$, then will x be found by this theorem.

$$\frac{4m^{\frac{5}{5}} + 3am^{\frac{4}{5}} + 2bm^{\frac{3}{5}} + cm^{\frac{2}{5}} - m}{d + 5m^{\frac{4}{5}} + 4am^{\frac{3}{5}} + 3bm^{\frac{2}{5}} + 2cm^{\frac{1}{5}}} = x,$$

In which if the value of $m^{\frac{1}{5}}$ being ſubſtituted, there is no remainder, the quotient gives one value of x; otherwiſe, this quotient muſt be ſubſtituted for $m^{\frac{1}{5}}$, and thus repeated till it either terminates or be as near the true root as neceſſary. The reaſon of thus aſſuming $m^{\frac{1}{5}}$ will be obvious, if we conſider, that m is always the product of all the roots (having their ſigns changed) and therefore $m^{\frac{1}{5}}$ will either be a root, or generally near one.

Example 1.

Required the five roots of this equation,

$$x^5 - 23x^4 + 159{\cdot}25x^3 - 459x^2 + 564{\cdot}75x - 243 = 0.$$

Here $m^{\frac{1}{5}} = 3$, which is affirmative becauſe the ſigns change alternately; and $4m^{\frac{5}{5}}\ (=4m) = +972$, $3am^{\frac{4}{5}} = -5589$, $2bm^{\frac{3}{5}} = +8599{\cdot}5$, $cm^{\frac{2}{5}} = -4131$, $-m = +243$; $d = +564{\cdot}75$, $5m^{\frac{4}{5}} = +405$, $4am^{\frac{3}{5}} = -2484$, $3bm^{\frac{2}{5}} = +4299{\cdot}75$, and $2cm^{\frac{1}{5}} = -2754$. Therefore

$$\frac{972 - 5589 + 8599{\cdot}5 - 4131 + 243}{564{\cdot}75 + 405 - 2484 + 4299{\cdot}75 - 2754} = \frac{+94{\cdot}5}{+31{\cdot}5} = +3,$$

a root. Now dividing the given equation by $x - 3$, we reduce it to this biquadratic, $x^4 - 20x^3 + 99{\cdot}25x^2 - 161{\cdot}25x + 81 = 0$; the roots of which will be found to be 1, 4, 13·5 and 1·5; hence the five roots of the propoſed equation are, 1, 1·5, 3, 4, and 13·5.

Example 2.

Given $x^5 + 15x^4 + 79x^3 + 189x^2 + 208x + 84 = 0$, required the values of x. Here $m^{\frac{1}{5}} = 2{\cdot}424$, &c. which as there are no changes in the ſigns muſt be negative, that is $m^{\frac{1}{5}} = -2{\cdot}424$. But in order to ſhorten the operation, either reject the decimals, or encreaſe them to unity; ſuppoſe the latter, then $m^{\frac{1}{5}} = -3$, and by proceeding as before we find

$$\frac{-972+3645-4266+1701-84}{405-1620+2133-1134+208} = \frac{+24}{-8} = -3;$$

which, as it terminates, is one root; and therefore by dividing the given equation by $x + 3$, we get the biquadratic $x^4 + 12x^3 + 43x^2 + 60x + 28 = 0$, whoſe roots are $-1, -2, -2, -7$, and thence $-1, -2, -2, -3$, and -7, are the roots of the propoſed equation.

Note. If in ſubſtituting the value of $m^{\frac{1}{5}}$ in the formula, the whole expreſſion ſhould vaniſh, or become equal 0, there will be two or more roots of the ſame value in the equation. As in this laſt Example by rejecting the decimals we have $m^{\frac{1}{5}} = -2$, from whence the formula becomes

$$\frac{-128 + 720 - 1264 + 756 - 84}{208 + 80 - 480 + 948 - 756} = \frac{0}{0};$$ hence there are two values of x, *viz.* -2, equal to each other. The reaſon of which will be evident to thoſe who are acquainted with the conſtruction of the theorem.

Example 3.

Given $x^5 - 7x^4 + 20x^3 - 155x^2 + 10000 = 0$. Required x. Here $a = -7$, $b = +20$, $c = -155$, $d = 0$, $-m = -10000$, and, by rejecting the decimals, $m^{\frac{1}{5}} = 6$, which may be aſſumed either affirmative or negative, if we take the latter, then a few operations, by ſubſtituting the quotient for $m^{\frac{1}{5}}$, give $x = 4\cdot54419572$, which is true to the 7th or 8th place.

Example 4.

Let the radius of a circle be 1, what is the length of the chord of 12 degrees?

The final equation will be $x^5 - 5x^3 + 5x - 1 = 0$. Here $a = 0$, $b = -5$, $c = 0$, $d = +5$, $m = +1$, and $m^{\frac{1}{5}} = 1$, and the formula gives $\frac{4-10+1}{5+5-15} = \frac{-5}{-5} = +1$, which is one root of the equation, but manifeſtly cannot be that anſwering the caſe propoſed; agreeable to what was obſerved in Rem. 2. Therefore aſſuming $m^{\frac{1}{5}} = \cdot2$, the firſt operation gives $m^{\frac{1}{5}} = \cdot209$, and the ſecond $m^{\frac{1}{5}} = \cdot209056926$, for the required value of x; or the length of the chord of 12 degrees, when the radius is unity, which is true to the laſt figure.

13. As it may be of ſome uſe to the young Algebraiſt, I ſhall now endeavour to illuſtrate, by a few Examples, the Newtonian method of obtaining the roots of literal equations.

Example 1.

Given, $y^3 + axy - a^2y - x^3 = 0$. Required the value of y, in a ſeries compoſed of the powers of a and x, with their coefficients.

The firſt thing to be done is to *tabulate* the equation, which is eaſily effected as follows: Make a right angle DAB (fig. 10.), and from A towards B write down all the powers of y, and from A towards D all the powers of x, as high as in the given equation, beginning from unity. Divide the whole into ſmall ſquares or parallelograms, as in the ſcheme, and inſert the terms of the equation in their correſponding ſquares or parallelograms, that is, in thoſe which have the ſame powers of x and y; then circumſcribe the ſignificant parallelograms with the polygon FBCDE, and the equation is tabulated or prepared for extraction.

Now, in order to determine the firſt term of the ſeries, make any two terms which are placed in two adjacent angles of the polygon equal to 0; that is,

$$\begin{aligned} y^3 - a^2y &= 0, \\ - a^2y - x^3 &= 0, \\ y^3 - x^3 &= 0, \end{aligned}$$

The two firſt equations give the firſt terms of a ſeries for y when x is ſuppoſed very little in reſpect to a; and the laſt, when a is little in reſpect to x. We ſhall take the firſt, *viz.*

$$y^3 = a^2y,$$

from whence $y = \pm a$; which is the firſt term of the ſeries.

To obtain the next term, put $y = a + p$, and ſubſtitute this value in the given equation, and we ſhall have,

$$\left.\begin{aligned} + y^3 &= a^3 + 3a^2p + 3ap^2 + p^3, \\ + axy &= a^2x + axp, \\ - a^2y &= - a^3 - a^2p, \\ - x^3 &= - x^3, \end{aligned}\right\} = 0$$

Select two terms of the greateſt value; or, which is the ſame thing, take two terms where p and x are ſeparately of the loweſt dimenſions, and make them equal o, from whence p will be had in terms of a and x. Here a, by ſuppoſition, is greater than x, and therefore much greater than p; conſequently $2a^2p + a^2x$ (that is, $3a^2p - a^2p + a^2x$) muſt be of greater value than all the other terms of the equation.

From $2a^2p = -a^2x$, we have $p = -\frac{x}{2}$; which is the ſecond term of the ſeries. Put $p = -\frac{x}{2} + q$, and ſubſtitute this value in the laſt equation, omitting all the terms which are higher than that power of x we propoſe the ſeries to be carried to, putting one term after the next leſs power of x, two after the next leſs, and ſo on. If we propoſe to carry the root to the fourth power of x, it will ſtand thus,

$$\left.\begin{array}{rrcl} & +p^3 &=& -\frac{x^3}{8} + \frac{3qx^2}{4}, \text{ \&c.} \\ & +3ap^2 &=& +\frac{3ax^2}{4} - 3aqx + 3aq^2, \\ 3a^2p - a^2p = & +2a^2p &=& -a^2x + 2a^2q, \\ & +axp &=& -\frac{ax^2}{2} + axq, \\ & +a^2x &=& +a^2x, \\ & -x^3 &=& -x^3, \end{array}\right\} = o.$$

Here the terms $\frac{ax^2}{4}\left(= \frac{3ax^2}{4} - \frac{ax^2}{2}\right)$ and $2a^2q$ are vaſtly greater than the other, q being very little in reſpect to x, and x very little in reſpect to a; therefore,

$$\frac{ax^2}{4} = -2a^2q;$$

from whence $q = -\frac{x^2}{8a}$, the third term of the ſeries.

Put $q = \frac{x^2}{8a} + r$, and ſubſtitute this value in the laſt equation, and we get,

$$\left.\begin{array}{r}
+3aq^2 = +\frac{3x^4}{6xa}, \text{ \&c.} \\
+2a^2q = -\frac{ax^2}{4} + 2a^2r, \\
+\frac{3x^2q}{4} = -\frac{3x^4}{32a}, \text{ \&c.} \\
-3axq + axq = -2axq = +\frac{x^3}{4} - 2axr, \\
\frac{3ax^2}{4} - \frac{ax^2}{2} = +\frac{ax^2}{4} = +\frac{ax^2}{4}, \\
-\frac{x^3}{8} - x^3 = -\frac{9x^3}{8} = -\frac{9x^3}{8},
\end{array}\right\} = 0.$$

The terms to be compared are $-\frac{7x^3}{8}\left(= +\frac{x^3}{4} - \frac{9x^3}{8}\right)$ and $+2a^2r$; that is,

$$2a^2r = \frac{7x^3}{8},$$

from whence $r = +\frac{7x^3}{16a^2}$, the fourth term of the ſeries.

Laſtly, let $r = +\frac{7x^3}{16a^2} + s$; this, ſubſtituted in the laſt equation, gives

$$\left.\begin{array}{l} -\,2axr = -\,\frac{7x^4}{8a},\ \&c. \\ +\,2a^2r = +\,\frac{7x^3}{8} + 2a^2s, \\ -\,\frac{7x^3}{8} = -\,\frac{7x^3}{8}, \\ -\,\frac{3x^4}{64a} = -\,\frac{3x^4}{64a}, \end{array}\right\} = 0.$$

Here all the terms vaniſh but $2a^2s$, and $-\frac{59x^4}{64a}$ $\left(= -\frac{7x^4}{8a} - \frac{3x^4}{64a}\right)$; which, being therefore compared as before, *viz.*

$$2a^2s = \frac{59x^4}{64a},$$

we have $s = +\frac{59x^4}{128a^3}$, the fifth term of the ſeries;

ſo $y = a - \frac{x}{2} - \frac{x^2}{8a} + \frac{7x^3}{16a^2} + \frac{59x^4}{128a^3}$, &c.

It will eaſily be perceived, by a proper attention to the method of operation, that every new term in the ſeries is had by dividing the loweſt of the terms affected with the indefinitely ſmall quantity x, or its powers, without the aſſumed ones, p, q, r, &c. by the quantity with which the aſſumed one of one dimenſion only is multiplied; thus, the ſecond term is had by dividing a^2x by $2a^2$; the third by dividing $\frac{ax^2}{4}$ by $2a^2$, &c. From which it is evident, that when the work is continued ſo far, as that the aſſumed quantity (p, q, r, &c.) is only of one dimenſion, the remaining terms of the root may be had by one diviſion; ſo, when $-\frac{x^2}{8a} + r$ is ſubſtituted for y, it appears that r is only of one dimenſion, therefore the remaining two terms will be had thus,

$$\begin{array}{rl}
2a^2 - 2ax\Big) & \frac{7x^3}{8} + \frac{3x^4}{64}\Big(+ \frac{7x^3}{16a^2} + \frac{59x^4}{128a^3}. \\
& \underline{\frac{7x^3}{8} - \frac{7x^4}{8a}} \\
& \quad + \frac{59x^4}{64a} \\
& \quad \underline{+ \frac{59x^4}{64a}, \text{ \&c.}}
\end{array}$$

Example 2.

Given $\frac{y^5}{5} - \frac{y^4}{4} + \frac{y^3}{3} - \frac{y^2}{2} + y + a - x = 0$. Required y in terms of a and x, which are here ſuppoſed to be nearly equal.

In order then to make the ſeries properly converge, we muſt ſubſtitute for their difference, that is, put $x = a + z$, and the given equation will become

$$\frac{y^5}{5} - \frac{y^4}{4} + \frac{y^3}{3} - \frac{y^2}{2} + y - z = 0.$$

This being tabulated (as in fig. 11.) we have $y = z$, for the firſt term of the ſeries. Put $y = z + p$, and write this value in the above equation, and there ariſes,

$$\left.\begin{array}{rl}
+ \frac{1}{5}y^5 = & * \\
- \frac{1}{4}y^4 = & - \frac{1}{4}z^4, \text{ \&c.} \\
+ \frac{1}{3}y^3 = & + \frac{1}{3}z^3 + z^2p \\
- \frac{1}{2}y^2 = & - \frac{1}{2}z^2 - zp - \frac{1}{2}p^2 \\
+ y = & + z + p \\
- z = & - z
\end{array}\right\} = 0.$$

The terms to be equated are p, and $-\frac{1}{2}z^2$, from which the ſecond term of the ſeries is $+\frac{1}{2}z^2$. Put $p = \frac{1}{2}z^2 + q$, and ſubſtitute this in the laſt equation for p, and it becomes,

$$\left.\begin{array}{lcl} -\frac{1}{2}p^2 & = & -\frac{1}{8}z^4, \text{ \&c.} \\ +z^2p & = & +\frac{1}{2}z^4, \text{ \&c.} \\ -zp & = & -\frac{1}{2}z^3 - zq \\ +p & = & +\frac{1}{2}z^2 + q \\ -\frac{1}{4}z^4 & = & -\frac{1}{4}z^4 \\ +\frac{1}{3}z^3 & = & +\frac{1}{3}z^3 \\ -\frac{1}{2}z^2 & = & -\frac{1}{2}z^2 \end{array}\right\} = 0.$$

Here the terms to be compared are q and $-\frac{z^3}{6}, \left(= -\frac{z^3}{2} + \frac{z^3}{3}\right)$ but as q is only of one dimenſion, the remaining terms may be found by diviſion, thus;

$$\begin{array}{l} 1 - z\Big) + \frac{z^3}{6} - \frac{z^4}{8}\Big(+ \frac{z^3}{6} + \frac{z^4}{24}. \\ \qquad\quad + \frac{z^3}{6} - \frac{z^4}{6} \\ \hline \qquad\qquad\quad + \frac{z^4}{24} \\ \qquad\qquad\quad + \frac{z^4}{24}, \text{ \&c.} \\ \hline \end{array}$$

Therefore the root is $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$, &c.; or, by reſtoring the value of x,

$$y = x - a + \frac{\overline{x-a}|^2}{2} + \frac{\overline{x-a}|^3}{3.2} + \frac{\overline{x-a}|^4}{3.4.2}, \text{ \&c.}$$

where the law is manifeſt, and may be continued at pleaſure, *viz.*

$$+ \frac{\overline{x-a}|^5}{3.4.5.2} + \frac{\overline{x-a}|^6}{3.4.5.6.2}, \text{ \&c.}$$

It appears from the table that other ſeries may be found from the equations,

$$\tfrac{1}{5}y^5 = z, \text{ and } \tfrac{1}{5}y^4 - \tfrac{1}{4}y^3 + \tfrac{1}{3}y^2 - \tfrac{1}{2}y = -1.$$

Remark. Sir Iſaac, or ſome of his tranſcribers, ſeem to have committed an overſight in ſuppoſing we might put either $a + z$, or $a - z$ for x, for it is indiſpenſably neceſſary that x ſhould be greater than a, otherwiſe the ſeries will not converge. The ſubſtituting of $a - z$ for x gives the laſt term of the given equation affirmative, and from thence the value of y comes out $z + 2z + 4z + 8z + 16z$, &c. which will evidently be a *diverging* ſeries, however ſmall we ſuppoſe z to be. Therefore, in order to have a *converging* ſeries for y, when a is greater than x, we muſt change the ſpecies, that is, write a for x, and x for a. So if $a - x = z$, a will be equal $z + x$; this ſubſtituted in the given equation, gives the laſt term negative as before, therefore the root will be

$$y = a - x + \frac{\overline{a-x}|^2}{2} + \frac{\overline{a-x}|^3}{3.2} + \frac{\overline{a-x}|^4}{3.4.2}, \text{ \&c.}$$

Example 3.

Given $y^3 + y^2 + y - x^3 = 0$. Required y in terms of x. Here it is evident that x muſt be very great, and therefore the common method cannot exhibit a true value of y. But to make the ſeries converge, put $\frac{1}{x} = z$, and thence will $x = \frac{1}{z}$. Subſtitute this for x in the given equation, and there ariſes $y^3 + y^2 + y - \frac{1}{z^3} = 0$. Tabulate this equation (fig. 12.) and the terms to be compared will be y^3 and $\frac{1}{z^3}$,

from

from whence $y = \frac{1}{z}$, the firſt term of the ſeries, and proceeding as before, the root is found

$$= \frac{1}{z} - \frac{1}{3} - \frac{2}{9}z + \frac{7}{81}z^2 + \frac{5}{81}z^3, \&c.$$

Reſtoring x it becomes $y = x - \frac{1}{3} - \frac{2}{9x} + \frac{7}{81x^2}$, &c.

The other equations are $y = \frac{1}{z^3}$, and $y^2 + y = -1$. The former gives $y = \frac{1}{z^3} - \frac{1}{z^6} + \frac{1}{z^9}$, &c. which, as it aſcends in the powers of z in the denominators, muſt alſo aſcend in the powers of x in the numerators; and therefore will evidently diverge; and the latter gives $y = \sqrt{-\frac{3}{4}} - \frac{1}{2}$, impoſſible.

Example 4.

Given $y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \frac{35}{1152}y^9 + \frac{63}{2816}y^{11}$, &c. $- x = 0$. Required x in terms of y. Here y, and conſequently x, muſt be ſuppoſed leſs than unity, in order that the ſeries may duly converge. Having tabulated the equation (fig. 13) we have $y = x$, the firſt term of the ſeries. Put $y = x + p$, and ſubſtitute this value of y in as many terms of the given equation as the ſeries is propoſed to be carried to; ſuppoſe to four terms, and the operation will ſtand as follows,

$$\begin{array}{rcl}
+\frac{63}{2816}y^{11} & = & * \\
+\frac{35}{1152}y^{9} & = & * \\
+\frac{5}{112}y^{7} & = & +\frac{5}{112}x^7, \&c. \\
+\frac{3}{40}y^{5} & = & +\frac{3}{40}x^5 + \frac{3}{8}x^4p, \&c. \\
+\frac{1}{6}y^{3} & = & +\frac{1}{6}x^3 + \frac{1}{2}x^2p + \frac{1}{2}xp^2, \&c. \\
+y & = & +x + p \\
 & & -x
\end{array}$$

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Fig. 10

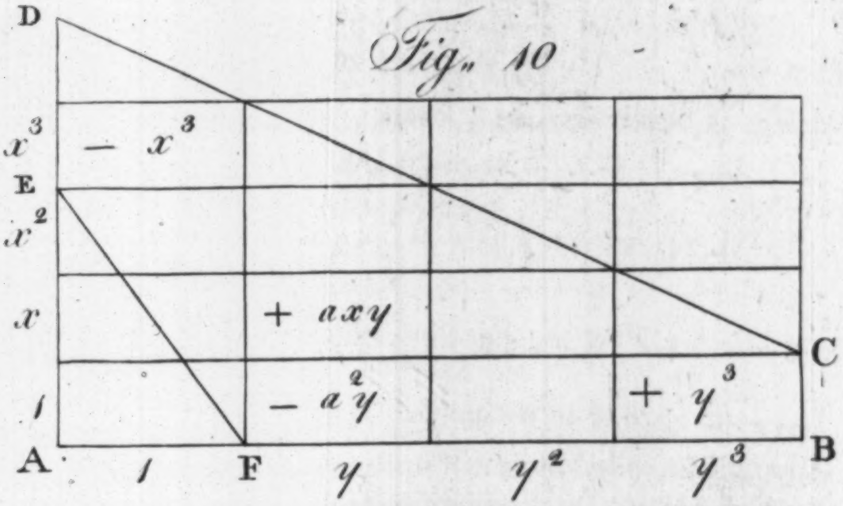


Fig. 11

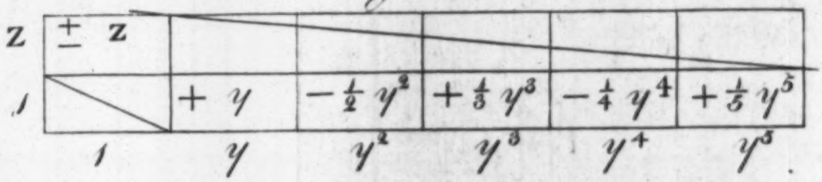


Fig. 12

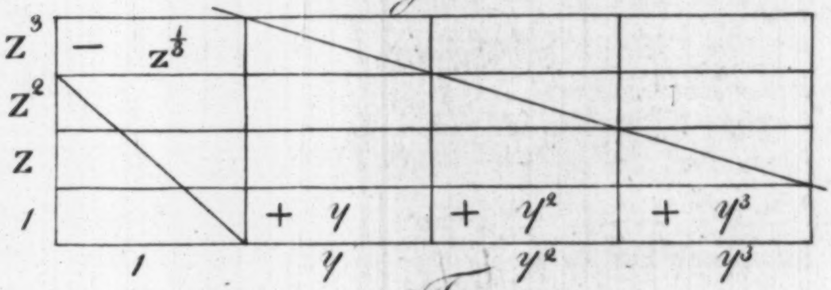
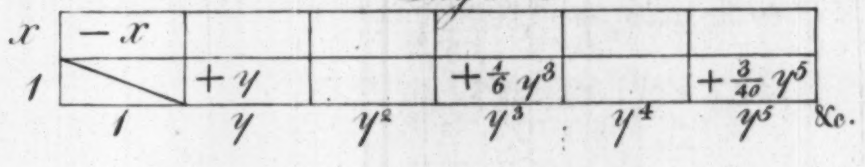


Fig. 13



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Hence $p = -\frac{1}{6}x^3$. Put $p = -\frac{1}{6}x^3 + q$, and ſubſtitute this in the laſt equation, and it becomes,

$$\left.\begin{array}{l} +\frac{1}{2}xp^2 = +\frac{1}{72}x^7, \text{ \&c.} \\ +\frac{3}{8}x^4p = -\frac{1}{16}x^7, \text{ \&c.} \\ +\frac{1}{2}x^2p = -\frac{1}{12}x^5 + \frac{1}{2}x^2q \\ + \quad p = -\frac{1}{6}x^3 + q \\ \qquad\qquad + \frac{5}{112}x^7 \\ \qquad\qquad + \frac{3}{40}x^5 \\ \qquad\qquad + \frac{1}{6}x^3 \end{array}\right\} = \left\{\begin{array}{l} -\frac{1}{252}x^7, \text{ \&c.} \\ \\ -\frac{1}{120}x^5 + \frac{1}{2}x^2q \\ \\ + q \end{array}\right\} = 0.$$

Here q is of one dimenſion only, therefore,

$$\begin{array}{l} 1+\frac{x^2}{2}\Big) + \frac{1}{120}x^5 + \frac{1}{252}x^7\Big(+ \frac{1}{120}x^5 - \frac{1}{5040}x^7, \text{ \&c.} \\ \qquad \underline{+\frac{1}{120}x^5 + \frac{1}{240}x^7} \\ \qquad\qquad\qquad -\frac{1}{5040}x^7 \\ \qquad\qquad\qquad \underline{-\frac{1}{5040}x^7, \text{ \&c.}} \end{array}$$

Hence the root is $y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$, &c.

or, $y = x - \frac{x^3}{6} + \frac{x^5}{4.5.6} - \frac{x^7}{4.5.6.7.6} + \frac{x^9}{4.5.6.7.8.9.6} - \frac{x^{11}}{4.5.6.7.8.9.10.11,6}$, &c, the law of continuation being evident.

This laſt example is uſually called the *reverſion of a ſeries*. There are ſeveral other methods of performing it; but this, I think, is as eaſy to be underſtood as any of them. The young ſtudent may here ſee with what facility theſe intricate affairs are managed when freed from all that unneceſſary prolixity we find them embarraſſed with in moſt authors. What before ſeemed almoſt inſuperable to him may be now only a pleaſant and agreeable exerciſe, as he cannot fail of underſtanding the method of operation, if he but attentively examine theſe examples. Indeed, there may be a great many literal equations propoſed which may not coincide with any of the examples we have given. But when the equation is tabulated, and two or three of the firſt terms of the ſeries obtained, it will immediately appear whether it properly converges or no. If all the equations diverge, or have impoſſible roots, a little artifice muſt be uſed, ſuch as augmenting or diminiſhing the roots by ſome known quantity; or by taking the reciprocal of an indefinitely large quantity, and ſo on. So that in almoſt all caſes the ſeries may be made to converge, and the root obtained by the foregoing method.

Some uſeful Remarks on the nature and method of Fluxions.

As I have always found the following remarks of ſervice to the learner, I ſhall make no apology for inſerting them in this treatiſe, which, I have before obſerved, is purely deſigned as an help to the young reader, in removing ſome of thoſe little obſtacles which he muſt unavoidably meet with at his firſt entrance on theſe ſtudies.

The doctrine of *prime and ultimate ratios*, by which the fluxions of quantities are generally inveſtigated, or demonſtrated, contains in it ſomething ſo very obſcure and unintelligible to the learner, that it is rather more apt to confuſe than give a proper arrangement to his ideas on the ſubject *.

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* The firſt Lemma of Sir Iſaac Newton's Principia appears to many to be very exceptionable; his words are, —*Quantitates, ut et quantitatum rationes, quæ ad æqualitatem tempore quovis finito conſtanter tendunt, et ante finem temporis illius propius ad invicem accedunt quam pro data quavis differentia, fiunt ultimo æquales.* And then adds, *Si negas; fiant ultimo inæquales, et ſit earum ultima differentia* D. *Ergo nequeunt propius ad æqualitatem accedere quam pro data differentia* D: *contra hypotheſin.* Now, though this method of demonſtration is far from being ſatisfactory, being a kind of reaſoning *mutatis mutandis*; yet, it is plain that the poſition may be readily admitted if the *decrements* only, or *parts deſtroyed*, are to be proportional. For inſtance, ſuppoſe two lines, the one 20 inches, the other 12, to be diminiſhed by ſome cotemporary decrements, of which the ratio is reſpectively as 3 to 1. Here then it is evident, that the quantities 20 and 12, would ſoon become equal by ſuch a diminution. Thus, in the firſt portion of time, let the former line be reduced to 14; then the latter will become 10. In the ſecond portion of time, the former becomes 8, and alſo the latter 8. They have therefore converged to equality by a cotemporary diminution with proportional decrements. But when the quantities themſelves are ſuppoſed to be proportionably diminiſhed, and thereby to obtain the ratio of equality, I think it will appear that there is nothing more abſurd. For ſuppoſe the quantities 20 and 12 to be any-how proportionably diminiſhed, ſuppoſe by a continual biſection, and it is evident, that they will converge to equality in reſpect of their difference, and yet retain their original proportion. Thus, the firſt difference of the propoſed quantities 20 and 12, is 8 inches; the firſt biſection reduces that difference to 4 inches, and the quantities themſelves to 10 and 6; but 10 is to 6, as 20 is to 12; therefore the proportion is not altered. The ſecond biſection makes the difference only 2 inches; but ſtill the quantities are in the ſome ratio, for 5 is to 3, as 20 to 12. The third biſection reduces the difference to 1 inch, and the quantities themſelves to 2 ½ and 1 ½, which are ſtill in the ſame proportion. Hence then it plainly appears, that two quantities may converge to equality in reſpect of their difference, and that *that* difference may become leſs than any aſſignable quantity, yet the quantities themſelves can never become equal; for in whatever ratio they were originally, in the ſame ratio will they remain, if

The moſt natural and eaſy way of acquiring a right notion of fluxions, is by the introducing of *time* into the account. For by this means we do not conſider them as mere velocities, which

diminiſhed *ſine ſine*, according to the mathematical doctrine of the infinite diviſibility of matter.

How abſurd then muſt it appear, to attempt to find the laſt ratio of the cotemporary increments of two flowing quantities by a continual diminution of thoſe increments, ſince it is obvious they will always remain in the ſame ratio, however ſmall they are taken. And yet after this manner have Colſon, Ditton, Hayes, and ſeveral other writers on fluxions, pretended to find the laſt ratio of the vaniſhing increments. Thus if the increment of x be denoted by $\acute{x}$, the cotemporary increment of x^n will be expreſſed by $nx^{n-1}\acute{x} + \frac{n^2-n}{2}x^{n-2}\acute{x}^2 + \frac{n^3-3n^2+2n}{2.3}x^{n-3}\acute{x}^3$, &c. Here then, ſay they, by continually diminiſhing theſe increments, we approach nearer to the ratio with which they firſt ariſe or begin to be generated; and therefore, when $\acute{x}$ is become infinitely ſmall, the higher powers of it muſt vaniſh firſt, and leave the infinitely ſmall increments in the ratio of $\acute{x}$ to $nx^{n-1}\acute{x}$; which is therefore their *prime* or *ultimate ratio*. But does this appear in the leaſt ſatisfactory? Is it not rather a mere quibble? For if the increment of x have a *real* value, though ever ſo ſmall, it is obvious, that the increment of x^n cannot be accurately $nx^{n-1}\acute{x}$. And if $\acute{x}^2$, $\acute{x}^3$, &c. be abſolutely *nothing*, certainly the root itſelf muſt be *nothing* alſo; and conſequently the whole expreſſion muſt vaniſh together. Beſides, this is plainly contradictory to the lemma; for, if that be admitted, the laſt ratio muſt be that of equality and not of 1 to nx^{n-1}, ſince the quantities are proportionably diminiſhed in the ſame time.

Some other writers, who, I imagine, ſaw the fallacy of this method of reaſoning, have endeavoured to obviate theſe difficulties, by repreſenting the affair in a different light, as follows; If we conſider that the increment of x^n is much different from the fluxion of it, the former being deſcribed by an accelerated motion, and the latter by an uniform one, it will not be ſo hard to conceive, that by continually diminiſhing the increment of the ſimple quantity x, the increment of the compound quantity x^n will come

nearer

which naturally involve the mind in metaphyſical difficulties; but as the magnitudes they uniformly generate in a given finite time, ſuppoſing the fluent or ſpace to be deſcribed by an uniform motion. And if the motion by which any magnitude is generated be not uniform, but accelerated or retarded, the idea of a fluxion will ſtill be the ſame: For though we cannot expreſs the fluxion by any ſpace *actually* generated in a given time, as in uniform motion; yet we can readily aſſign the magnitude, or (as it is commonly called) the cotemporary increment, that *would be* uniformly generated, if the acceleration or retardation were to ſtop at any point in which the fluxion is required to be inveſtigated.

Now, as our ideas of magnitude ariſe from a compariſon of the propoſed object with ſome other of determinate dimenſions: ſo, in the method of fluxions, we fix on a given magnitude, which is ſuppoſed to have been uniformly generated in a given time by the motion of a point, line, or plane, as a ſtandard,

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nearer to an uniform velocity; and therefore, juſt in the inſtant of vaniſhing, or when it becomes *nothing*, the velocity muſt be uniform, and truly expreſs the fluxion at that point. Hence then it appears, that it is by confounding the increment and fluxion, that theſe ſeeming abſurdities ariſe. For if we imagine the fluxion of the *uniformly* generated quantity to be an infinitely ſmall, yet certain magnitude, it will readily appear, by keeping our ideas of the fluxion and increment diſtinct, how the ratio of the fluxions may be had, when the increments themſelves vaniſh, or become *nothing*. Thus, let $\dot{x}$ denote the fluxion of the ſimple quantity x, or that infinitely ſmall quantity which would be uniformly deſcribed by the generating point in one inſtant of time; then, ſince the ratio of the increments, $\acute{x} : nx^{n-1}\acute{x} + \frac{n^2-n}{2}x^{n-2}\acute{x}^2$, &c. is the ſame as $1 : nx^{n-1} + \frac{n^2-n}{2}x^{n-1}\acute{x}$, &c. or, by multiplying by $\dot{x}$, as $\dot{x} : nx^{n-1}\dot{x} + \frac{n^2-n}{2}x^{n-2}\dot{x}\acute{x}$, &c. it will be evident, that if we ſuppoſe the increment of x to vaniſh, the ratio will then become accurately, $\dot{x} : nx^{n-}\dot{x}$. But this is too metaphyſical for moſt readers.

wherewith to compare any other magnitude, which is ſuppoſed to have been generated in the ſame time, by an accelerated or retarded motion. Thus (for the ſake of illuſtration) ſuppoſe a ball to roll on an horizontal plane, in a ſtraight direction, at the uniform rate of 20 feet in a minute; and alſo another ball to move uniformly in the ſame direction, at the rate of 40 feet in the ſame time; here then it will be plain, that the magnitudes generated in any given time muſt be in the ratio of 2 to 1; and therefore the fluxion of the latter will be double that of the former. And from hence it appears, that if the fluxion of x be $\dot{x}$, that of $2x$ will be $2\dot{x}$, $3x$ will be $3\dot{x}$, &c. and generally, that of nx will be $n\dot{x}$. But if, while one ball moves along with an uniform velocity, the other is ſuppoſed to move with an accelerated motion, and that the law of acceleration is ſuch, that the ſpace deſcribed by the latter, from the commencement of motion, is always ſome power of that deſcribed by the former, ſuppoſe the ſquare of it; then, if the magnitude by which the ſpace that is uniformly deſcribed is increaſed in a given time be denoted by $\dot{x}$, that magnitude which the accelerated motion would uniformly generate in the ſame time, and commencing from the ſame inſtant, will be expreſſed by $2x\dot{x}$. Thus, in the caſe propoſed, if the firſt ball has uniformly deſcribed a ſpace of 10 poles, the other muſt have run 100 poles; but the former ball moves uniformly at the rate of 20 feet in a minute, therefore the magnitude or ſpace, which the accelerated ball would uniformly deſcribe from the ſame inſtant in one minute, will be 400 feet. The fluxions will be therefore at that point in the ratio of 400 to 20; or of 20 to 1. This is eaſily demonſtrated as follows: Let $x - t$, and x, repreſent the ſpaces deſcribed by the uniform motion at any two given times, then (by Hypotheſis) will the cotemporary ſpaces, paſſed over by the accelerated motion, be $(\overline{x - t}|^2 =) \; x^2 - 2xt + t^2$; and x^2; and therefore $x^2 - \overline{x^2 - 2xt + t^2} = 2xt - t^2$ will be the difference of thoſe ſpaces. From whence it follows, that, while the firſt ball runs uniformly over the ſpace t, the other runs over the ſpace $2xt - t^2$. Now, as this ſpace is not generated by an uniform but accelerated motion, it cannot repreſent the fluxion of either of the ſpaces expreſſed by $\overline{x - t}|^2$ or x^2; but it may evidently repreſent the magnitude which might be uniformly deſcribed with the mean velocity at ſome point between $\overline{x - t}|^2$ and x^2. But this magnitude is to that generated in the ſame time by the uniform motion as $2xt - t^2$ is to t; or, by multiplying by $\frac{\dot{x}}{t}$ as $2x\dot{x} - t\dot{x}$ to $\dot{x}$; therefore, when $x - t$ by the uniform motion of the ball becomes x, t vaniſhes, or is equal to *nothing*, and conſequently the point of mean velocity then coincides with x; and hence the above ratio becomes barely, as $2x\dot{x} : \dot{x}$. Therefore, according to the common phraſe, the fluxion of x^2 is $2x\dot{x}$.

This may be generally repreſented thus: Let the law of acceleration (or retardation, prefixing the negative ſign) be univerſally expreſſed by x^n, and by the ſame method of reaſoning, by help of the binomial theorem, we ſhall find the cotemporary fluxions to be in the ratio of $nx^{n-1}\dot{x}$ to $\dot{x}$. For if $x - t$ and x repreſent the ſpaces uniformly deſcribed as before, the cotemporary ſpaces deſcribed by the accelerated motion will be expreſſed by $\overline{x - t}|^n$ and x^n; but the difference of theſe ſpaces $x^n - \overline{x - t}|^n$ is equal $nx^{n-1}t - \frac{n^2 - n}{2} . x^{n-2}t^2$, &c. therefore the ratio of the magnitudes, generated in the ſame time by the uniform motion and the mean velocity, will be expreſſed by $nx^{n-1}t - \frac{n^2 - n}{2} . x^{n-2}t^2$, &c. to t; or, by multiplying by $\frac{\dot{x}}{t}$, as $nx^{n-1}\dot{x} - \frac{n^2 - n}{2} . x^{n-2}t\dot{x}$, &c. to $\dot{x}$;

and when $x-t$ is equal x, t will evidently be $=0$, hence all the terms wherein t is found muſt vaniſh, and the point of mean velocity coincide with x; conſequently the ratio of the fluxions will be as $nx^{n-1}\dot{x}$ to $\dot{x}$.

Hence then it appears, that we have the moſt rational notion of fluxions from the conſideration of time in the generation of the increment or decrement, and that the fluxion of any variable quantity may be truly defined, *The magnitude by which any flowing quantity would be increaſed in a given time with the generating velocity at a given inſtant, ſuppoſing it from thence to proceed uniformly or invariably.* And with regard to the higher orders of fluxions, how much more obſcure are our notions without the idea of time in the operation of the fluent generating the increment; ſince by having recourſe to the firſt ratio of the naſcent increment, or the laſt ratio of the evaneſcent increment, even to obtain only the firſt fluxion of a variable quantity, we unavoidably fall into this abſurdity, *That a velocity which continues for no time at all actually deſcribes a ſpace.* How then can we form any conception not only of ſuch a ſpace or increment, but alſo of an infinite variety of magnitudes of it, generated in one and the ſame point and inſtant of time, in which it is well known all the orders of fluxions are conſidered, when nothing, I think, can be more evident than that the magnitude or increment imagined to be generated muſt in ſuch a caſe be *purum putum nihil*, or ſtrictly and abſolutely nothing. If a doubt of the exiſtence of an increment under ſuch circumſtances be deemed incredulity and a ſpecies of infidelity*, I am afraid I ſhall be ſtigmatized with thoſe appellations; for I confeſs it is paſt my comprehenſion how a mere point can contain in itſelf an infinite variety of magnitudes, and which are all at the ſame time equal to one another. Theſe unneceſſary quibbles, and metaphyſical niceties, by which ſome have attempted to explain the principles of fluxions, have not only rendered them quite obſcure to the learner, but alſo expoſed them to the ridicule and ſevere criticiſms of ſeveral writers of great abilities in the mathematics. But theſe criticiſms, it is probable, were not intended to invalidate the method of fluxions (which it is evident may be ſtrictly mathematically demonſtrated) but to ſhew the futility of the method they had taken to elucidate the principles; in which light it is well known the incomparable inventor never intended they ſhould be viewed.

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* See Colſon's Newton's Flux. p. 18, Preface.

From what has been ſaid we may draw theſe practical obſervations.

1. That the common rule for finding the fluxion of a flowing quantity, *viz. Multiply the fluxion of the root by the exponent of the power and the affixed coefficient, and the product by that power of the ſame root of which the exponent is leſs by unity than the given exponent*, is general, and without exceptions, being applicable to any expreſſion whatever conſiſting of one variable quantity with a conſtant exponent. 2. If the expreſſion be a compound one, that is a binomial, trinomial, or any multinomial, the fluxion of each term muſt be found ſeparately, and connected with their reſpective reſulting ſigns; the ſum ariſing by ſuch addition is the fluxion of the compound expreſſion. 3. If the expreſſion conſiſts of the product of two or more variable quantities, each quantity muſt flow ſeparately, while the others are ſuppoſed to be conſtant, or as coefficients to that variable quantity; the ſum of theſe fluxions will be that of the given expreſſion. This follows from the general expreſſion $nx^{n-1}\dot{x}$. Thus, let the fluxion of yz be propoſed to be inveſtigated. Put $y+z=v$, then will $y^2+2yz+z^2=v^2$;

hence $yz = \frac{v^2 - y^2 - z^2}{2} = \frac{1}{2}v^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2$. And, from what has been before ſhewn, the fluxion of this will be $v\dot{v} - y\dot{y} - z\dot{z}$; but $v = y + z$, and $\dot{v} = \dot{y} + \dot{z}$, $\because$ by ſubſtitution, the fluxion of yz is $y\dot{z} + z\dot{y}$. And in the ſame manner will the fluxion of xyz be found to be $xy\dot{z} + xz\dot{y} + yz\dot{x}$; and that of $uxyz = uxy\dot{z} + uxz\dot{y} + uyz\dot{x} + xyz\dot{u}$; &c.

We ſhall now give a few examples in order to ſhew the propriety of theſe remarks.

Let the fluxion of x be $\dot{x}$; that is, let the magnitude or ſpace which is paſſed over in a given time by the uniform motion of a point in generating the ſpace x be denoted by $\dot{x}$; then will the fluxions, or relative magnitudes, which would be uniformly generated in the ſame time, of the cotemporary fluents, or ſpaces already paſſed over when the fluxions are compared, be obtained by the general rule as follows,

Variable Quantities.	*Fluxions.*
1. $2x$.	$1 \times 2x^{1-1}\dot{x} = 2\dot{x}$ *.
2. nx,	$1 \times nx^{1-1}\dot{x} = n\dot{x}$.
3. x^3,	$3x^{3-1}\dot{x} = 3x^2\dot{x}$.
4. $5x^7$,	$7 \times 5x^{7-1}\dot{x} = 35x^6\dot{x}$.
5. mx^n,	$mnx^{n-1}\dot{x}$.
6. $\frac{m}{n}x^{\frac{t}{r}}$,	$\frac{t}{r} \times \frac{m}{n}{}^{\frac{t}{r}-1}\dot{x}, = \frac{tm}{rn}x^{\frac{t-r}{r}}\dot{x}$.
7. $\frac{m}{n}x^{\frac{t}{r}}y^{\frac{p}{q}}$,	$y^{\frac{p}{q}} \times \frac{m}{n} \times \frac{t}{r}x^{\frac{t}{r}-1}\dot{x} + x^{\frac{t}{r}} \times \frac{m}{n} \times \frac{p}{q}y^{\frac{p}{q}-1}\dot{y}, = \frac{mt}{nr}y^{\frac{p}{q}}x^{\frac{t-r}{r}}\dot{x} + \frac{mp}{nq}x^{\frac{t}{r}}y^{\frac{p-q}{q}}\dot{y}$.

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* That x^{1-1} is equal to 1, will be evident, if we conſider that x^{1-1} is $= x^1 \times x^{-1}$; which is equal $\frac{x^1}{x^1} = 1$. And therefore, generally, $x^{n-n} = \frac{x^n}{x^n} = 1$.

Variable

Variable Quantities.	*Fluxions.*
8. $\frac{x}{y}, = xy^{-1},$	$x \times -y^{-2}\dot{y} + y^{-1}\dot{x}, = \frac{y\dot{x}-x\dot{y}}{yy}.$
9. $\frac{1}{\sqrt[n]{x}}, = x^{-\frac{1}{n}},$	$-\frac{1}{n}x^{-\frac{1+n}{n}}\dot{x}, = -\frac{\dot{x}}{nx^{\frac{1+n}{n}}}.$
10. $\sqrt{a^2-x^2}, = \overline{a^2-x^2}\rvert^{\frac{1}{2}},$	$\frac{1}{2}\cdot\overline{a^2-x^2}\rvert^{-\frac{1}{2}} \times -2x\dot{x}, = -\frac{x\dot{x}}{\sqrt{a^2-x^2}}.$
11. $a\sqrt{x^2+y^2}, = a.\overline{x^2+y^2}\rvert^{\frac{1}{2}},$	$\frac{1}{2}a \times \overline{x^2+y^2}\rvert^{-\frac{1}{2}} \times \overline{2x\dot{x}+2y\dot{y}}, = \frac{ax\dot{x}+ay\dot{y}}{\sqrt{x^2+y^2}}.$
12. $\sqrt[\frac{m}{n}]{y}, = y^{\frac{n}{m}}$ *,	$\frac{n}{m}y^{\frac{n-m}{m}}\dot{y}, = \frac{n\dot{y}}{my^{\frac{m-n}{m}}}.$

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* It may, perhaps, appear ſtrange to the young reader, that $\sqrt[\frac{m}{n}]{y}$ ſhould be $= y^{\frac{n}{m}}$. But, if he is acquainted with the nature of logarithms, he need not be informed, that to extract the $\frac{m}{n}$th root is to divide the log. of y by $\frac{m}{n}$; but it is eaſily proved that to divide by $\frac{m}{n}$ is the ſame as to multiply by its reciprocal $\frac{n}{m}$; therefore $\sqrt[\frac{m}{n}]{y} = y^{\frac{n}{m}}$. Thus, if $m = 1$, and $n = 2$; $\sqrt[\frac{1}{2}]{y}$ will be equal y^2. For dividing the log. of y by $\frac{1}{2}$, is evidently the ſame as multiplying it by 2.

Variable

Variable Quantities.	*Fluxions.*
13. $\sqrt[\frac{m}{n}]{yx}, = y^{\frac{n}{m}} x^{\frac{n}{m}},$	$\frac{nx^{\frac{n}{m}}\dot{y}}{my^{\frac{m-n}{m}}} + \frac{ny^{\frac{n}{m}}\dot{x}}{mx^{\frac{m-n}{m}}}.$
14. $\sqrt{y^m z^n}, = y^{\frac{m}{2}} z^{\frac{n}{2}},$	$\frac{n\sqrt{y^m}\dot{z}}{2z^{\frac{2-n}{2}}} + \frac{m\sqrt{z^n}\dot{y}}{2y^{\frac{2-m}{2}}}.$
15. $\frac{m}{n}\sqrt[\frac{r}{t}]{x^{\frac{p}{q}} + y^{\frac{v}{w}}},$	$\frac{\frac{p}{q}x^{\frac{p-q}{q}}\dot{x} + \frac{v}{w}y^{\frac{v-w}{w}}\dot{y}}{\frac{tn}{rm}\sqrt[\frac{r}{r-t}]{x^{\frac{p}{q}} + y^{\frac{v}{w}}}}.$
16. $z.\sqrt[n]{x^m y^n + z^t},$	$\frac{x^m y^n z^{n-1}\dot{z} + x^m z^n y^{n-1}\dot{y} + \frac{m}{n}y^n z^n x^{m-1}\dot{x} + \frac{t+n}{n}z^{t+n-1}\dot{z}}{\sqrt[\frac{n}{n-1}]{\overline{x^m y . z}^n + z^{t+n}}}.$

Variable Quantities.	*Fluxions.*

And by the ſame rule are the fluxions of logarithms found, only obſerving to put down the number itſelf under the fluxion, as a denominator.

17. $L.\sqrt[q]{\overline{x^m+y^n}}$,	$\dfrac{\frac{1}{q}\overline{x^m+y^n}\rvert^{\frac{1-q}{q}}\times\overline{mx^{m-1}\dot{x}+ny^{n-1}\dot{y}}}{\sqrt[q]{\overline{x^m+y^n}}}=\dfrac{mx^{m-1}\dot{x}+ny^{n-1}\dot{y}}{q\times\overline{x^m+y^n}}$.
18. $L^n.\overline{x+y}\rvert^m$.	$nL^{n-1}.\overline{x+y}\rvert^m\times\dfrac{m.\overline{x+y}\rvert^{m-1}\times\overline{\dot{x}+\dot{y}}}{\overline{x+y}\rvert^m}=nL^{n-1}.\overline{x+y}\rvert^m\times\dfrac{m\dot{x}+m\dot{y}}{x+y}$.

And from hence may the fluxion of any quantity, of which the index is variable, be eaſily found.

Thus, the fluxion of A^z is $A^z L.A\dot{z}$. For if we put $A^z = v$, then will $\dot{v}$ = the fluxion of A^z. Take the logarithm on both ſides, and we have $L.A^z$ = (from the nature of logarithms) $zL.A =$ $L.v$, $\because \dot{z}L.A=\frac{\dot{v}}{v}$, and $vL.A\dot{z}=\dot{v}$; hence $A^zL.A\dot{z}$ is the fluxion of A^z.

If

If the quantity and index are both variable, the fluxion may be had after the ſame manner. Suppoſe

If the quantity and index are both variable, the fluxion may be had after the ſame manner. Suppoſe the fluxion of $\overline{x-y}|^{z^{v}}$ was required, where all the quantities are variable. The operation will ſtand thus, $\overline{x-y}|^{z^{v}} = w$; $z^{v}\mathrm{L}.\overline{x-y} = \mathrm{L}.w$; $vz^{v-1}\dot{z}\,\mathrm{L}.\overline{x-y} + z^{v}\mathrm{L}.z\dot{v}\,\mathrm{L}.\overline{x-y} + z^{v} \times \frac{\dot{x}-\dot{y}}{x-y} = \frac{\dot{w}}{w}$;

$\therefore \overline{x-y}|^{z^{v}} \times \overline{vz^{v-1}\dot{z}\,\mathrm{L}.\overline{x-y} + z^{v}\mathrm{L}.z\dot{v}\,\mathrm{L}.\overline{x-y} + z^{v} \times \frac{\dot{x}-\dot{y}}{x-y}}$, is the fluxion of $\overline{x-y}|^{z^{v}}$.

In the *inverſe method of fluxions*, nothing can be more eaſy than to proceed in a retrograde order from the fluxion to the fluent (in expreſſions that will admit of it) by the converſe of the general rule, namely, *Strike out the fluxionary letter, add* one *to the index, and divide by the index ſo increaſed.*

Example.

Required the fluent of $nx^{n-1}\dot{x}$. The proceſs will ſtand thus; $nx^{n-1}\dot{x}$, nx^{n-1}, nx^{n-1+1}, nx^{n}, $\frac{nx^{n}}{n}$, x^{n}, the fluent.

In expreſſions affected with a radical, if the fluxion without the radical, or vinculum, be the fluxion of the quantity under it, the fluent may be obtained by the general rule, obſerving to ſtrike out, not the fluxionary letter *only*, but *all* that part which appears to be the fluxion of the quantity under the radical.

Example.

Required the fluent of $\sqrt[\frac{1}{n}]{a^{m} + x^{m}} \times mx^{m-1}\dot{x}$. The proceſs will be as follows;

$\overline{a^{m} + x^{m}}\big|^{n} \times mx^{m-1}\dot{x}$, $\overline{a^{m} + x^{m}}\big|^{n}$, $\overline{a^{m} + x^{m}}\big|^{n+1}$, $\frac{\overline{a^{m} + x^{m}}\big|^{n+1}}{n + 1}$

the fluent.

If the fluxion without the radical be not that of the quantit under it, but in a *given* ratio to it, the fluent may ſtill be ha by the general rule; obſerving to increaſe or decreaſe it in tha given ratio.

Example.

Required the fluent of $\overline{a + bx^{m}}\big|^{n} \times cx^{m-1}\dot{x}$.

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The fluxion of bx^m is $bmx^{m-1}\dot{x}$; therefore the ratio of the fluxion of the quantity under the radical is to that without it, as $bm : c$. Hence the proceſs will be thus;

$$\overline{a+bx^m}\rceil^n \times cx^{m-1}\dot{x},\ \overline{a+bx^m}\rceil^n,\ \overline{a+bx^m}\rceil^{n+1},\ \frac{\overline{a+bx^m}\rceil^{n+1}}{n+1};$$

then, as $bm : c :: \frac{\overline{a+bx^m}\rceil^{n+1}}{n+1} : \frac{\overline{a+bx^m}\rceil^{n+1} \times c}{bm \times \overline{n+1}}$, the fluent.

If the fluxion without the radical be neither the fluxion of the quantity contained under it, nor in any given ratio to it, yet, in many forms of expreſſions, the fluents may ſtill be had in finite terms, by help of the binomial theorem, which will always terminate, when a number equal to the exponent of the flowing quantity without the radical comes to be ſubtracted from it.

Example.

Required the fluent of $x^n\dot{x}\sqrt[m]{a+x}$.

Here the ratio of the fluxion of the quantity under the radical is to that without it, as $1 : x^n$, and conſequently not given; therefore the eaſieſt way is to proceed by ſubſtitution, as follows; put $a+x=v$, then will $x=v-a$, $x^n=\overline{v-a}\rceil^n$, and $\dot{x}=\dot{v}$, $\therefore\ x^n\dot{x}\sqrt[m]{a+x}=v^{\frac{1}{m}}\dot{v}\times\overline{v-a}\rceil^n$. But by the binomial theorem, $\overline{v-a}\rceil^n=v^n-nv^{n-1}a+\frac{n^2-n}{2}v^{n-2}a^2$, &c. So that when n is a whole number, it is evident the ſeries will terminate, and the fluent be had in finite terms. Suppoſe $n=2$, and $m=3$, then the fluxion will become $v^{\frac{7}{3}}\dot{v}-2av^{\frac{4}{3}}\dot{v}+a^2v^{\frac{1}{3}}\dot{v}$, and the fluent by the common rule is $\frac{3}{10} v^{\frac{10}{3}} - \frac{6}{7} a v^{\frac{7}{3}} + \frac{3}{4} a^2 v^{\frac{4}{3}}$; which, by reſtoring the value of v, becomes $\frac{3}{10} . \overline{a + x}|^{\frac{10}{3}} + \frac{3a^2}{4} . \overline{a + x}|^{\frac{4}{3}} - \frac{6a}{7} . \overline{a + x}|^{\frac{7}{3}}$, the required fluent.

And when the flowing part of the quantity under the radical is a power, it always follows by ſubſtitution, that if the exponent of the flowing quantity without the radical be an even number, the fluent may be had in finite terms; but if an odd number, it will run into an infinite ſeries, becauſe the index of the binomial then becomes a fraction.

Example.

Required the fluent of $x^n \dot{x} \sqrt[m]{a^2 + x^2}$.

Put $a^2 + x^2 = v$, then will $x = \sqrt{v - a^2}$, $x^n = \overline{v - a^2}|^{\frac{n}{2}}$, and $\dot{x} = \frac{\dot{v}}{2x} = \frac{\dot{v}}{2\sqrt{v - a^2}}$; $\therefore$ $x^n \dot{x} \sqrt[m]{a^2 + x^2} = \frac{v^{\frac{1}{m}} \dot{v} \times \overline{v - a^2}|^{\frac{n}{2}}}{2\sqrt{v - a^2}} = \frac{1}{2} v^{\frac{1}{m}} \dot{v} \times \overline{v - a^2}|^{\frac{n-1}{2}}$. From whence it appears, that if n be an odd number, the index will be an integer, and the ſeries will terminate; but if it be an even number, the index will be a fraction, and the ſeries will run on *ſine ſine*. Suppoſe $n = 5$, and $m = 2$, then the expreſſion becomes $\frac{1}{2} v^{\frac{1}{2}} \dot{v} \times \overline{v - a^2}|^2$; which being expanded, the fluent of each term taken by the common rule, and the value of v reſtored, gives $\frac{1}{7} . \overline{a^2 + x^2}|^{\frac{7}{2}} + \frac{1}{3} . \overline{a^2 + x^2}|^{\frac{3}{2}} - \frac{2}{5} a^2 . \overline{a^2 + x^2}|^{\frac{5}{2}}$ for the required fluent.

If the flowing part of the quantity under the radical be raiſed to ſome power, and the fluxion without it be only that of the root, the fluent muſt be obtained by infinite ſeries. This, indeed, may be often avoided; for if the given fluxion can be reduced to a form ſimilar to that of the arch of a circle, then the length of that arch, which is always expreſſed in terms of either the ſine, verſed ſine, tangent, or ſecant, and the radius, will be the required fluent.

Example.

Required the fluent of $\frac{c^2\dot{y}}{\sqrt{4c^2-4y^2}}$.

The expreſſion $\frac{c^2\dot{y}}{\sqrt{4c^2-4y^2}}$ is $= \frac{cc\dot{y}}{2\sqrt{cc-yy}} = \frac{1}{2}c \times \frac{c\dot{y}}{\sqrt{cc-yy}}$; but $\frac{c\dot{y}}{\sqrt{cc-yy}}$ appears to be the fluxion of the arch of a circle, of which the radius is c, and ſine y; and therefore when the degrees in the arch are known, the length of the arch is known alſo, which being multiplied by $\frac{1}{2}c$ gives the fluent of $\frac{c^2\dot{y}}{\sqrt{4c^2-4y^2}}$.

And when it happens that we cannot reduce the fluxion to any of thoſe forms, yet, if we can by any reduction diſcover, that the numerator of the expreſſion is the fluxion of the denominator (which frequently occurs in the ſolutions of problems) then will the hyperbolic logarithm of the denominator be the fluent required.

Example.

Required the fluent of $\frac{c^3\dot{x}}{c^2-x^2}$.

The expreſſion $\frac{c^3\dot{x}}{c^2-x^2}$ is $=\frac{1}{2}c^2\times\frac{2c\dot{x}}{c^2-x^2}$; and $\frac{2c\dot{x}}{c^2-x^2}$ is $=$ $\frac{2c\dot{x}}{\overline{c+x}\times\overline{c-x}}=\frac{2c\dot{x}}{\overline{c-x}|^2}\times\frac{c-x}{c+x}$; but this expreſſion is the ſame as $\frac{2c\dot{x}}{\overline{c-x}|^2}\div\frac{c+x}{c-x}$, or $\frac{2c\dot{x}\times\overline{c-x}|^{-2}}{\overline{c+x}\times\overline{c-x}|^{-1}}$, where the numerator is evidently the fluxion of the denominator; therefore the hyp. log. of $\frac{c+x}{c-x}$ being multiplied by $\frac{1}{2}c^2$ will be the required fluent.

But as it would be rather foreign to our preſent purpoſe, and ſwell the book beyond its intended ſize, to endeavour to explain all the various methods of finding fluents by a proper number of examples, we ſhall therefore only give one or two more of reducing ſurd fluxionary expreſſions by infinite ſeries, and Emerſon's tables; which will ſerve as a ſpecimen of the method of operation with all other ſimilar forms. Suppoſe we ſelect one from Emerſon's Trigonometry *p.* 27, ſecond Edit. where he gives $\frac{r\dot{y}}{\sqrt{rr-yy}}$ for the fluxion of the arch of a circle.

Firſt Method.

$\frac{r\dot{y}}{\sqrt{rr-yy}}=\frac{\sqrt{r^2-y^2}\times r\dot{y}}{r^2-y^2}$, ∴ therefore the reduction will ſtand as follows.

$$\begin{array}{rl}
r^2-y^2\Big(& r-\frac{y^2}{2r}-\frac{y^4}{8r^3}-\frac{y^6}{16r^5}-\frac{5y^8}{128r^7}, \text{ \&c.} \\
 & \underline{r^2} \\
2r-\frac{y^2}{2r}\Big) & -y^2 \\
 & \underline{-y^2+\frac{y^4}{4r^2}} \\
2r-\frac{y^2}{r}-\frac{y^4}{8r^3}\Big) & -\frac{y^4}{4r^2} \\
 & \underline{-\frac{y^4}{4r^2}+\frac{y^6}{8r^4}+\frac{y^8}{64r^6}} \\
2r-\frac{y^2}{r}-\text{\&c.}\Big) & -\frac{y^6}{8r^4}-\frac{y^8}{64r^6} \\
 & \underline{-\frac{y^6}{8r^4}+\frac{y^8}{16r^6}+\text{\&c.}} \\
2r-\text{\&c.}\Big) & -\frac{5y^8}{64r^6}-\text{\&c.} \\
 & \underline{-\frac{5y^8}{64r^6}+\text{\&c.}}
\end{array}$$

And the root multiplied by $r\dot{y}$ gives

$$r^2\dot{y}-\frac{y^2\dot{y}}{2}-\frac{y^4\dot{y}}{8r^2}-\frac{y^6\dot{y}}{16r^4}-\frac{5y^8\dot{y}}{128r^6}, \text{ \&c.}$$

this divided by r^2-y^2 will be in this form, *viz.*

$$r^2 - y^2)\,r^2\dot{y} - \frac{y^2\dot{y}}{2} - \frac{y^4\dot{y}}{8r^2} - \frac{y^6\dot{y}}{16r^4} - \frac{5y^8\dot{y}}{128r^6} \text{ \&c.}\left(\dot{y} + \frac{y^2\dot{y}}{2r^2} + \frac{3y^4\dot{y}}{2.4r^4} + \frac{5y^6\dot{y}}{2.4.2r^6} + \frac{7.5y^8\dot{y}}{2.4.2.8r^6}, \text{ \&c.}\right.$$

$$\underline{r^2\dot{y} - y^2\dot{y}}$$

$$+ \frac{y^2\dot{y}}{2} - \frac{y^4\dot{y}}{8r^2}$$

$$\underline{+ \frac{y^2\dot{y}}{2} - \frac{y^4\dot{y}}{2r^2}}$$

$$+ \frac{3y^4\dot{y}}{8r^2} - \frac{y^6\dot{y}}{16r^4}$$

$$\underline{+ \frac{3y^4\dot{y}}{8r^2} - \frac{3y^6\dot{y}}{2.4r^4}}$$

$$+ \frac{5y^6\dot{y}}{2.4.2r^4} - \frac{5y^8\dot{y}}{128r^6}$$

$$\underline{+ \frac{5y^6y}{2.4.2r^4} - \frac{5y^8\dot{y}}{2.4.2r^6}}$$

$$+ \frac{7.5y^8\dot{y}}{2.4.2.8r^6} - \text{\&c.}$$

$$\underline{+ \frac{7.5y^8\dot{y}}{2.4.2.8r^6} - \text{\&c.}}$$

$$\therefore \frac{r\dot{y}}{\sqrt{r^2-y^2}} = \dot{y} + \frac{y^2\dot{y}}{2r^2} + \frac{3y^4\dot{y}}{2.4r^4} + \frac{5y^6\dot{y}}{2.4.2r^6} + \frac{7.5y^8\dot{y}}{2.4.2.8r^8}, \text{\&c.}$$

and the fluent is

$$y + \frac{y^3}{3.2r^2} + \frac{3y^5}{5.2.4r^4} + \frac{5y^7}{7.2.4.2r^6} + \frac{5.7y^9}{9.2.4.2\ 8r^8}, \text{\&c.}$$

$$\text{or, } y + \frac{y^3}{3.2r^2} + \frac{3y^5}{5.2.4r^4} + \frac{3.5y^7}{7.2.4.6r^6} + \frac{3.5.7y^9}{9.2.4.6.8r^8}, \text{\&c.}$$

Second Method.

$$\frac{r\dot{y}}{\sqrt{r^2-y^2}} = \sqrt{r^2-y^2} \times r\dot{y} \times \frac{1}{r^2-y^2}; \text{ that is,}$$

$$r^2\dot{y} - \frac{y^2\dot{y}}{2} - \frac{y^4\dot{y}}{8r^2} - \frac{y^6\dot{y}}{16r^4} - \frac{5y^8\dot{y}}{128r^6}, \text{\&c.} \times : \frac{1}{r^2} + \frac{y^2}{r^4} + \frac{y^4}{r^6} + \frac{y^6}{r^8}, \text{\&c.}$$

Multiply each term of the firſt ſeries into all the terms of the latter, rejecting thoſe which exceed the power you intead carrying it to, and it will ſtand thus,

$$\begin{array}{rrrrrl}
\dot{y} & + \frac{y^2\dot{y}}{r^2} & + \frac{y^4\dot{y}}{r^4} & + \frac{y^6\dot{y}}{r^6} & + \frac{y^8\dot{y}}{r^8}, & \text{\&c.} \\
 & - \frac{y^2\dot{y}}{2r^2} & - \frac{y^4\dot{y}}{2r^4} & - \frac{y^6\dot{y}}{2r^6} & - \frac{y^8\dot{y}}{2r^8}, & \text{\&c.} \\
 & & - \frac{y^4\dot{y}}{8r^4} & - \frac{y^6\dot{y}}{8r^6} & - \frac{y^8\dot{y}}{8r^8}, & \text{\&c.} \\
 & & & - \frac{y^6\dot{y}}{16r^6} & - \frac{y^8\dot{y}}{16r^8}, & \text{\&c.} \\
 & & & & - \frac{5y^8\dot{y}}{128r^8}, & \text{\&c.} \\
\hline
\dot{y} & + \frac{y^2\dot{y}}{2r^2} & + \frac{3y^4\dot{y}}{2.4r^4} & + \frac{5y^6\dot{y}}{2.4.2r^6} & + \frac{7.5y^8\dot{y}}{2.4.2.8r^8}, & \text{\&c.}
\end{array}$$

from whence the fluent is had as before.

Third Method.

$$\frac{r\dot{y}}{\sqrt{r^2-y^2}} = r \times \overline{r^2-y^2}\rceil^{-\frac{1}{2}} \times \dot{y},$$

By Form 16, in Emerſon's excellent Treatiſe of Fluxions, we have $a = r^2$, $\beta = -1$, $z = y$, $\eta = 2$, $\frac{\mu}{\nu} = -\frac{1}{2}$; (or $\mu = -1$, $\nu = 2$,) and $\pi = 0$; then by ſubſtitution

$$\frac{a^{\frac{\mu}{\nu}} z^{\pi+1}}{\pi+1} = \frac{r^{2\times-\frac{1}{2}} y^{0+1}}{0+1} = \frac{r^{-1}y}{1} = \frac{y}{r},$$

$A = r^{-1}y$, $q = -\frac{y^2}{r^2}$, ∵

$$\frac{\frac{\mu}{\nu}Aq}{\pi+1+\eta} = \frac{-\frac{1}{2}\times\frac{y}{r}\times-\frac{y^2}{r^2}}{0+1+2} = \frac{y^3}{3\ 2r^4},$$

$B = +\frac{y^3}{2r^3}$, ∴

$$\frac{\frac{\mu-\nu}{2\nu}Bq}{\pi+1+2\nu} = \frac{\frac{-1-2}{4}\times\frac{y^3}{2r^3}\times-\frac{y^2}{r^2}}{0+1+4} = \frac{3y^5}{5\cdot2.4r^5},$$

$C = +\frac{3y^5}{2.4r^5}$, ∴

$$\frac{\frac{\mu-2\nu}{3\nu}Cq}{\pi+1+3\nu} = \frac{\frac{-1-4}{6}\times\frac{3y^5}{2.4r^5}\times-\frac{y^2}{r^2}}{0+1+6} = \frac{3\cdot5y^7}{7.2.4.6r^7}, \text{ \&c.}$$

Therefore the formula gives

$$\frac{y}{r} + \frac{y^3}{3\cdot2r^3} + \frac{3y^5}{5.2.4r^5} + \frac{3\cdot5y^7}{7.2.4.6r^7}, \text{ \&c.}$$

for the fluent of $\frac{\dot{y}}{\sqrt{r^2-y^2}}$; which being multiplied by r, we have $y + \frac{y^3}{3\cdot2r^2} + \frac{3y^5}{5\cdot2\ 4r^4}$, &c. as before.

It alſo appears, from the conſtruction of the fluxion, that the fluent will be expreſſed by the arch of a circle, of which the rad. is unity, and ſine $\frac{y}{r}$; which may be very conveniently had from the Tables of Nat. Sines, &c. thus; Eind the degrees, &c. correſponding to the given value of $\frac{y}{r}$, which call d; then it will be, as 180 : 3·14159 :: d : $\frac{3{\cdot}14159d}{180}$ the required fluent.

This one example indeed might be ſufficient to ſhew the method of procedure with all ſuch expreſſions; but, in order to make it as plain as poſſible, we ſhall take another example, which is more complex, from *p.* 30 of the ſame book, *viz.* $\frac{r\dot{v}}{\sqrt{2rv-vv}}$; being the fluxion of the arch of a circle, expreſſed in terms of the radius and verſed ſine.

$$\frac{r\dot{v}}{\sqrt{2rv-vv}} = \frac{\sqrt{2rv-vv}}{2rv-vv} \times r\dot{v},\ \therefore$$

$2rv-v^2$

$$\begin{array}{rl}
 & \underline{\frac{2rv-v^2}{2rv}}\left(\sqrt{2rv}-\frac{v^2}{2\sqrt{2rv}}-\frac{v^4}{2\cdot4\overline{\sqrt{2rv}}\big|^3}-\frac{v^6}{2\cdot2\cdot4\overline{\sqrt{2rv}}\big|^5},\ \&c.\right. \\
2\sqrt{2rv}-\frac{v^2}{2\sqrt{2rv}}\Big) & -v^2 \\
 & \underline{-v^2+\frac{v^4}{4\overline{\sqrt{2rv}}\big|^2}} \\
2\sqrt{2rv}-\frac{v^2}{\sqrt{2rv}},\ \&c.\Big) & \quad -\frac{v^4}{4.2rv} \\
 & \quad \underline{-\frac{v^4}{4.2rv}+\frac{v^6}{2.4\overline{\sqrt{2rv}}\big|^4},\ \&c.} \\
2\sqrt{2rv},\ \&c.\Big) & \qquad -\frac{v^6}{2.4.\overline{2rv}\big|^2},\ \&c. \\
 & \qquad \underline{-\frac{v^6}{2.4.\overline{2rv}\big|^2},\ \&c.}
\end{array}$$

$2rv-vv$

$$2rv-vv)\sqrt{2rv}-\frac{v^2}{2\sqrt{2rv}}-\frac{v^4}{2.4\sqrt{\overline{2rv}|^3}}-\frac{v^6}{4.4\sqrt{\overline{2rv}|^5}},\ \&c.\left(\frac{\sqrt{2rv}}{2rv}+\frac{\sqrt{2rv}.v^2}{2.\overline{2rv}|^2}+\frac{3.\sqrt{2rv}.v^4}{4.2.\overline{2rv}|^3}+\frac{15.\sqrt{2rv}.v^6}{6.4.2.\overline{2rv}|^4},\ \&c.\right.$$

$$\sqrt{2rv}-\frac{v\sqrt{2rv}}{2r}$$

$$+\frac{v^2\sqrt{2rv}}{2.2rv}-\frac{v^4}{2.4\sqrt{\overline{2rv}|^3}}$$

$$+\frac{v^2\sqrt{2rv}}{2.2rv}-\frac{\sqrt{2rv}.v^4}{2.\overline{2rv}|^2}$$

$$+\frac{3.\sqrt{2rv}.v^4}{4.2.\overline{2rv}|^2}-\frac{v^6}{4.4\sqrt{\overline{2rv}|^5}},\ \&c.$$

$$+\frac{3\sqrt{2rv}.v^4}{4.2.\overline{2rv}|^2}-\frac{3.\sqrt{2rv}.v^6}{4.2.\overline{2rv}|^3},\ \&c.$$

$$+\frac{3.5\sqrt{2rv}.v^6}{6.4.2.\overline{2rv}|^3},\ \&c.$$

$$+\frac{3.5\sqrt{2rv}.v^6}{6.4.2.\overline{2rv}|^3},\ \&c.$$

The quotient multiplied by $r\dot{v}$ gives

$$\frac{\sqrt{2rv}\,r\dot{v}}{2rv}+\frac{\sqrt{2rv}.v^2r\dot{v}}{2.\overline{2rv}|^2}+\frac{3\sqrt{2rv}.v^4r\dot{v}}{4.2.\overline{2rv}|^3}+\frac{3.5\sqrt{2rv}.v^6r\dot{v}}{6.4.2.\overline{2rv}|^4}, \text{ \&c.}$$

or, $\frac{1}{2}\sqrt{2r}.v^{-\frac{1}{2}}\dot{v}+\frac{1}{2}\times\frac{\sqrt{2r}.v^{\frac{1}{2}}\dot{v}}{2^2r}+\frac{3}{4.2}\times\frac{\sqrt{2r}.v^{\frac{3}{2}}\dot{v}}{2^3.r^2}+\frac{3.5}{6.4.2}$
$\times\frac{\sqrt{2r}.v^{\frac{1}{2}}v^2\dot{v}}{2^4.r^3}$, &c.

And the fluent is,

$$\sqrt{2rv}+\frac{\sqrt{2rv}.v}{2^2.3r}+\frac{3.\sqrt{2rv}.v^2}{2^3.4.5r^2}+\frac{3.5\sqrt{2rv}.v^3}{2^4.4.6.7r^3}, \text{ \&c.}$$

or, $\sqrt{2rv}\times:1+\frac{v}{2^2.3r}+\frac{3v^2}{2^3.4.5r^2}+\frac{3.5v^3}{2^4.4.6.7r^3}$, &c.

The fluent may alſo be had by Form 16, Emerſon's Fluxions, as follows:

$\frac{r\dot{v}}{\sqrt{2rv-vv}}=r\times\overline{2r-v}|^{-\frac{1}{2}}\times v^{-\frac{1}{2}}\dot{v}$, therefore, $a=2r$, $\beta=-1$, $z=v$, $\eta=1$, $\pi=-\frac{1}{2}$, and $\frac{\mu}{\nu}=-\frac{1}{2}$; or $\mu=-1$, and $\nu=2$.

And by ſubſtitution,

$$\frac{a^{\frac{\mu}{\nu}}z^{\pi+1}}{\pi+1}=\frac{\overline{2r}|^{-\frac{1}{2}}v^{\frac{1}{2}}}{\frac{1}{2}}=\frac{2\sqrt{v}}{\sqrt{2r}},$$

$\text{A}=\frac{\sqrt{v}}{\sqrt{2r}}$; $q=-\frac{v}{2r}$, ∴

$$\frac{\frac{\mu}{\nu}\,\text{A}q}{\pi+1+\eta}=\frac{-\frac{1}{2}\times\frac{\sqrt{v}}{\sqrt{2r}}\times-\frac{v}{2r}}{-\frac{1}{2}+1+1}=\frac{2v^{\frac{1}{2}}v}{3.2^2r\sqrt{2r}},$$

B =

$$B = \frac{vv^{\frac{1}{2}}}{2^2 r\sqrt{2r}},\ \therefore$$

$$\frac{\frac{\mu - \nu}{2\nu}\, Bq}{\pi + 1 + 2n} = \frac{\frac{-1-2}{4} \times \frac{vv^{\frac{1}{2}}}{2^2 r\sqrt{2r}} \times -\frac{v}{2r}}{\frac{5}{2}} = \frac{2.3v^2v^{\frac{1}{2}}}{\sqrt{2r}.4.2^3.5r^2},$$

$$C = \frac{3v^2v^{\frac{1}{2}}}{4.2^3r^2\sqrt{2r}},\ \therefore$$

$$\frac{\frac{\mu - 2\nu}{3\nu}\, Cq}{\pi \times 1 \times 3\nu} = \frac{\frac{-1-4}{6} \times \frac{3v^2v^{\frac{1}{2}}}{4.2^3r^2\sqrt{2r}} \times -\frac{v}{2r}}{-\frac{1}{2} + 1 + 3} = \frac{2.3.5v^3v^{\frac{1}{2}}}{2^4.4.6.7r^3\sqrt{2r}}.$$

&c.

Hence we have

$$\frac{2\sqrt{v}}{\sqrt{2r}} + \frac{2v^{\frac{1}{2}}v}{3.2^2r\sqrt{2r}} + \frac{2.3v^2v^{\frac{1}{2}}}{5.4.2^3r^2\sqrt{2r}} + \frac{2.3.5v^3v^{\frac{1}{2}}}{7.4.6.2^4r^3\sqrt{2r}},\ \&c.$$

for the fluent of $\frac{\dot{v}}{\sqrt{2rv - vv}}$. Multiply by r, and it becomes

$$\frac{2rv^{\frac{1}{2}}}{\sqrt{2r}} + \frac{2rv^{\frac{1}{2}}v}{2^2.3r\sqrt{2r}} + \frac{3.2rv^{\frac{1}{2}}v^2}{2^3.4.5r^2\sqrt{2r}} + \frac{3.5.2rv^{\frac{1}{2}}v^3}{2^4.4.6.7r^3\sqrt{2r}},\&c.$$

or, $\sqrt{2rv} + \frac{\sqrt{2rv}.v}{2^2.3r} + \frac{\sqrt{2rv}.3v^2}{2^3.4.5r^2} + \frac{\sqrt{2rv}.3.5.v^3}{2^4.4.6.7r^3}$, &c.

for the fluent of $\frac{r\dot{v}}{\sqrt{2rv - vv}}$, as before.

Or thus,

$\frac{r\dot{v}}{\sqrt{2rv - vv}} = \frac{rv^{-\frac{1}{2}}\dot{v}}{\sqrt{2r - v}}$. This correſponds to Form 10, where $z = v$, $a = 2r$, $-\beta = -1$, (or $\beta = 1$,) and $n = 1$; and the fluent is, $2Nr \times$ Degrees of the arch of a circle, of which the radius is 1, and natural ſine $\sqrt{\frac{v}{2r}}$. Or, putting d for the degrees, it will be,

as $180 : 3{\cdot}14159 :: 2dr : \frac{3{\cdot}14159 \times 2dr}{180}$, the fluent.

As theſe examples will be ſufficient to explain the method of obtaining fluents by infinite ſeries, I ſhall therefore only farther obſerve to the learner, that, in the ſolutions of queſtions, the fluent firſt found generally wants correcting, by the addition or ſubtraction of ſome conſtant quantity; which is always to be determined by the nature of the queſtion, and may be effected by this eaſy rule: Subſtitute for the variable quantity in the fluent firſt found, that particular value which it is known to have when the *whole* fluent is ſuppoſed to be equal nothing; then, if the reſulting quantity be affirmative, ſubtract it from the fluent before found, but if negative add it; and it will be truly corrected. If the whole expreſſion ſhould vaniſh, the fluent needs no correction. An example or two will make it plain. Thus, in the fluxionary expreſſion $\dot{y} \times \overline{a+y}|^{4}$, the fluent firſt found is $\frac{\overline{a+y}|^{5}}{5}$. Now ſuppoſe when the *whole* fluent which this ought to expreſs is equal to nothing, that y is alſo equal to nothing, then will $\frac{\overline{a+y}|^{5}}{5}$ become $+\frac{a^{5}}{5}$; hence $\frac{\overline{a+y}|^{5}}{5}$ exceeds the whole fluent by $\frac{a^{5}}{5}$, which being therefore ſubtracted from it, leaves $\frac{\overline{a+y}|^{5} - a^{5}}{5}$ for the correct fluent. For if we put z for the whole fluent, then ought this equation $z = \frac{\overline{a+y}|^{5}}{5}$ to hold good in all the cotemporary values of z and y. But it appears that when $z = 0$, if we ſuppoſe $y = 0$ alſo, the expreſſion for the fluent, inſtead of vaniſhing, remains equal to $\frac{a^5}{5}$; this quantity therefore muſt evidently be either ſubtracted from $\frac{\overline{a+y}|^5}{5}$, or added to z to preſerve the equality. From whence the rule is obvious.

If when the *whole* fluent is ſuppoſed equal to nothing, y ſhould be known, from the nature of the problem, to have a certain value, ſtill the correction will be performed in the ſame manner. Thus, when the *whole* fluent is equal nothing, let $y = b$; then the above fluent corrected will be $\frac{\overline{a+y}|^5 - \overline{a+b}|^5}{5}$. As this is a very material point in the uſe of fluxions, it may not be amiſs to illuſtrate it by the ſolutions of a few queſtions, where the fluents will require a correction.

1. Required the area of a curve, of which the equation is $x^4 + a^2y^2 = a^2x^2$.

By reduction we have $y = \frac{x \times \sqrt{a^2 - x^2}}{a}$; multiply both ſides by the fluxion of the abciſs, and we get $\frac{\overline{a^2 - x^2}|^{\frac{1}{2}} \times x\dot{x}}{a}$ $= y\dot{x} =$ the fluxion of the area; the fluent of which found by the common rule, is $- \frac{\overline{a^2 - x}|^{\frac{3}{2}}}{3a}$. Now when the area, or whole fluent, is ſuppoſed to be $= 0$, it is evident that the abciſs x muſt be alſo $= 0$; hence the above expreſſion becomes $- \frac{a^2}{3}$, and therefore $\frac{a^2}{3} - \frac{\overline{a^2 - x}|^{\frac{3}{2}}}{3a}$ will be the correct fluent, or true value of the area.

2. Let the equation of a curve be $x^2y^2 + a^2y^2 = a^4$, required the area.

By reducing the equation we get $y = \frac{a^2}{\sqrt{a^2+x^2}}$; hence $\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}$ is the fluxion of the area. And becauſe $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ is $= \frac{\frac{\dot{x}}{\sqrt{a^2+x^2}} \times \overline{\sqrt{a^2+x^2}+x}}{\sqrt{a^2+x^2}+x}$, where the numerator is evidently the fluxion of the denominator, therefore the fluent is $a^2 \times$ hyp. log. of $x + \sqrt{a^2+x^2}$. Now put $x = 0$, then the fluent will become $a^2 \times$ hyp. log. of a, $\therefore a^2 \times$ hyp. log. of $\overline{x + \sqrt{a^2+x^2}} - a^2 \times$ hyp. log. of $a = a^2 \times$ hyp. log. of $\frac{x+\sqrt{a^2+x^2}}{a}$ is the correct value of the area.

3. To find the ſuperficies of an hyperbolic Conoid.

Put the ſemitranſverſe of the generating hyperbola $= t$, the ſemiconjugate $= c$, and the diſtance of any ordinate from the center $= x$ then by the property of the curve we have $y = \frac{c}{t}\sqrt{x^2-t^2}$, and therefore $\dot{y} = \frac{cx\dot{x}}{t\sqrt{x^2-t^2}}$; hence the fluxion of the area will be (putting $3{\cdot}141592 = p$) $\frac{2pc\dot{x}}{t^2} \times \sqrt{\overline{t^2+c^2} \times x^2 - t^4}$; or $\left(\text{by putting } a^2 = \frac{t^4}{t^2+c^2}\right)$ $\frac{2pc\dot{x}}{a}\sqrt{x^2-a^2}$. Now ſince the fluxion without the radical is not in any given ratio to that of the quantity under it, but is the fluxion of the root only, therefore, in order to avoid an infinite ſeries for the fluent, let the variable part of the fluxion be changed to $\frac{\dot{x} \times x^2 - a^2}{\sqrt{x^2-a^4}}$, which is equal to $\frac{x^3\dot{x} - \frac{1}{2}a^2x\dot{x}}{\sqrt{x^4-a^2x^2}} - \frac{\frac{1}{2}a^2x\dot{x}}{\sqrt{x^4-a^2x^2}}$. Here then the numerator of the firſt term is in a given ratio to the fluxion of the quantity under the radical, whence we get $\frac{pcx\sqrt{x^2-a^2}}{a}$ for the fluent of that term; from which ſubtracting the fluent of the other term $= \frac{\frac{1}{2}a^2\dot{x}}{\sqrt{x^2-a^2}}$, there ariſes $\frac{pcx\sqrt{x^2-a^2}}{a} - pca \times$ hyp. log. of $\overline{x+\sqrt{x^2-a^2}}$. But from the nature of the problem it is evident, that when the area is ſuppoſed to be $= 0$, that x will then be $= a$; therefore the correct fluent will be

$$\frac{pcx}{a}\sqrt{x^2-a^2} - pc^2 - pcd \times \text{hyp. log. of } \frac{x+\sqrt{x^2-a^2}}{t+\frac{ac}{t}} =$$

the true area of the ſuperficies.

Before we cloſe this ſubject, it may not be amiſs to give a ſhort Explanation of Trigonometrical Fluxions. Theſe ſhould by all means be well underſtood by the young Mathematician, on account of their great uſefulneſs in Aſtronomy, Navigation, Dialling, &c. For ſince the places of the heavenly bodies, the times of their riſing, ſetting, &c. their right aſcenſion, declination, latitude, longitude, amplitude, azimuth, &c. are all calculated by ſpherical triangles, of which the ſides and angles are variouſly affected by parallax, refraction, preceſſion of the equinoxes, obliquity of the ecliptic, &c. it is obvious, that where accuracy is required, it is neceſſary to make a correction of the variable parts, by ſuppoſing a ſmall continued motion in one or more of the great circles by which the triangle is formed, and thence by the fluxionary increaſe or decreaſe of the ſides and angles to determine the quantities ſought.

This ſubject naturally divides itſelf into four caſes as follow:

1. When an angle and a ſide adjacent are conſtant. 2. When an angle and the ſide oppoſite are conſtant. 3. When two of the ſides are conſtant. 4. When two of the angles are conſtant.

CASE I. Fig. 14.

In any ſpherical triangle ABC, if the angle B and a ſide adjacent to this angle, ſuppoſe BA, be conſtant or invariable, then ſhall we have the following analogies,

1. $\dot{\overline{BC}} : \dot{\overline{AC}} :: R : \text{coſ.}\,C.$

2. $\dot{\overline{BC}} : \dot{A} :: \text{ſin.}\,AC : \text{ſin.}\,C.$

3. $\dot{\overline{AC}} : \dot{A} :: \text{ſin.}\,AC \times \text{coſ.}\,C : R \times \text{ſin.}\,C :: \text{ſin.}\,AC : \text{tang.}\,A.$

4. $\dot{\overline{BC}} : \dot{C} :: \text{tang.}\,AC : \text{ſin.}\,C.$

That is,

1. As the fluxion of the variable ſide adjacent to the conſtant angle is to the fluxion of the ſide oppoſite, ſo is radius to the coſine of the angle oppoſite to the conſtant ſide.

2. As the fluxion of the variable ſide adjacent to the conſtant angle is to the fluxion of the angle oppoſite to this ſide, ſo is the ſine of the ſide oppoſite to the conſtant angle to the ſine of the angle oppoſite to the conſtant ſide.

3. As the fluxion of the ſide oppoſite to the conſtant angle is to the fluxion of the variable angle adjacent to the conſtant ſide, ſo is the rectangle under the ſine of this ſide and the coſine of the third angle to the rectangle under the ſine of the ſame angle and the radius, ſo is the ſine of the ſide oppoſite to the conſtant angle to the tangent of the angle oppoſite to the conſtant ſide.

4. As the fluxion of the ſide adjacent to the conſtant angle is to the fluxion of the angle adjacent to this ſide, ſo is the tangent of the ſide oppoſite to the conſtant angle to the ſine of the angle oppoſite to the conſtant ſide.

In order to give a clear demonſtration of theſe analogies, it will be neceſſary to premiſe the following

LEMMA.

If from the three angles of any ſpherical triangle ABC (fig. 15.) taken as poles, there be formed another ſpherical triangle DEF, we ſhall have, $\dot{\overline{DE}} = \dot{A}$, $\dot{\overline{DF}} = \dot{C}$, $\dot{\overline{FE}} = \dot{A}$; and $\dot{D} = \dot{\overline{AC}}$, $\dot{E} = \dot{\overline{AB}}$, $\dot{F} = \dot{\overline{BC}}$. For it is demonſtrated by moſt writers on ſpherics, that each ſide of the new triangle ſo deſcribed will be the ſupplement of the angle which is at its pole, and each of its angles the ſupplement of that ſide of the triangle ABC, to which it is oppoſite; and it is well known that an arc and its ſupplement have the ſame ſine, coſine, tangent, &c. therefore the fluxions of the ſides DE, EF, and FD, of the triangle DEF will be reſpectively equal to the fluxions of the oppoſite angles in the triangle ABC; and the fluxions of the angles D, E, F, the ſame as thoſe of the oppoſite ſides AC, AB, BC.

Suppoſe now in the triangle ABC that the great circle of which AC is a part, by a motion round A as a pole, comes into the poſition A*C*, and the ſides BC, AC will become B*c*, C*c*, reſpectively, which produce till they be each 90 degrees, and from the pole A deſcribe the little circular arcs C*C*, E*E*; then will the indefinitely ſmall quantities C*c*, *Cc*, be the reſpective fluxions of the ſides BC, AC, and the arc E*E* the meaſure of the variation of the angle A; and becauſe the

triangle

triangle C*Cc*, which is right-angled at *C*, may be eſteemed right-lined on account of its ſmallneſs, the ſides will be proportional to the ſines of their oppoſite angles. But as the angle at *c* is eſſentially the ſame as the angle at *C*, and conſequently the angle *c*C*C* the complement of that angle, we ſhall evidently have, C*c* : *Cc* :: R : coſ.C; or, $\dot{\overline{BC}} : \dot{\overline{AC}}$ :: R : coſ. C, the firſt analogy. 2. Becauſe the ſmall arcs E*E*, C*C*, contain the ſame number of degrees, it will be, E*E* : C*C* :: R : ſin. AC; and in the triangle C*Cc*, it is, C*C* : C*c* :: ſin. C : R, hence E*E* × C*C* : C*c* × C*C* :: R × ſin. C : R × AC, that is, E*E* : C*c* :: ſin. C : ſin. AC; or, $\dot{A} : \dot{\overline{BC}}$ :: ſin. C : ſin. AC, and by inverſion, $\dot{\overline{BC}} : \dot{A}$ :: ſin. AC : ſin. C, the ſecond analogy. 3. By inverſion, and multiplying the correſponding terms of the firſt and ſecond analogies, we get,

$\dot{\overline{AC}} : \dot{A}$ :: ſin. AC × coſ. C : ſin. C × R :: ſin. AC : tang. C, the third analogies. 4. And according to this caſe in the triangle DEF (fig. 15.) of which the parts are all ſupplements to thoſe of the triangle ABC, the angle at E and the adjacent ſide FE will be conſtant; and by the Lemma, the fluxions of the ſide DF and angle F will be reſpectively equal to the fluxions of the angle C and ſide BC of the triangle ABC; but by the laſt analogy it will be, $\dot{\overline{DF}} : \dot{F}$ :: ſin. DF : tang. D, and becauſe angles and their ſupplements have the ſame ſine and tangent, we have by ſubſtitution, $\dot{C} : \dot{\overline{BC}}$ :: ſin. C : tang. AC, and by inverſion, $\dot{\overline{BC}} : \dot{C}$ :: tang. AC : ſin. C, the fourth analogy. *Q. E. D.*

CASE II. Fig. 16.

When any one of the angles B and its oppoſite ſide AC are conſtant, we have theſe analogies,

1. $\dot{C} : \dot{\overline{AB}} :: \text{tang.}\,C : \text{tang.}\,AB.$

2. $\dot{A} : \dot{\overline{BC}} :: \text{tang.}\,A : \text{tang.}\,BC.$

3. $\dot{\overline{AB}} : \dot{\overline{BC}} :: \text{coſ.}\,C : \text{coſ.}\,A.$

4. $\dot{A} : \dot{C} :: \text{coſ.}\,BC : \text{coſ.}\,AB.$

That is,

1, 2. As the fluxion of either of the variable angles is to the fluxion of its oppoſite ſide, ſo is the tangent of this angle to the tangent of the ſame ſide.

3, 4. The fluxions of the variable ſides are as the coſines of their oppoſite angles; and the fluxions of the variable angles as the coſines of their oppoſite ſides.

DEMONSTRATION.

Since the ſin. AC : ſin. B :: ſin. AB : ſin. C : : ſin. BC : ſin. A, we have, $\text{ſin.}\,C = \frac{\text{ſin.}\,B}{\text{ſin.}\,AC} \times \text{ſin.}\,AB$, and $\text{ſin.}\,A = \frac{\text{ſin.}\,B}{\text{ſin.}\,AC} \times \text{ſin.}\,BC$; from whence it appears that the fluxions of ſin. C, and ſin. A, are as the fluxions of their oppoſite ſides AB and BC; and the fluxion of the arc which meaſures the angle C is to the fluxion of the arc AB as $\frac{\dot{\overline{\text{ſin.}\,C}}}{\text{coſ.}\,C}$ is to $\frac{\dot{\overline{\text{ſin.}\,AB}}}{\text{coſ.}\,AB}$*, or as $\dot{C} : \dot{\overline{AB}} :: \frac{\dot{\overline{\text{ſin.}\,C}}}{\text{coſ.}\,C} : \frac{\dot{\overline{\text{ſin.}\,AB}}}{\text{coſ.}\,AB}$. And becauſe the ſin. C has a conſtant ratio to ſin. AB, it will be, as $\dot{\overline{\text{ſin.}\,C}} : \dot{\overline{\text{ſin.}\,AB}} :: \text{ſin.}\,C : \text{ſin.}\,AB$; hence by ſuſtitution, $\dot{C} : \dot{\overline{AB}} :: \frac{\dot{\overline{\text{ſin.}\,C}}}{\text{coſ.}\,C} : \frac{\dot{\overline{\text{ſin.}\,AB}}}{\text{coſ.}\,AB}$, that is $\dot{C} : \dot{\overline{AB}} :: \text{tang.}\,C : \text{tang.}\,AB$, the firſt analogy. After the ſame manner it may be proved that, $\dot{A} : \dot{\overline{BC}} :: \text{tang.}\,A : \text{tang.}\,BC$, the ſecond analogy. And if D*A* be ſuppoſed equal to DA, D*C* equal to DC, and the angle at D very ſmall; the angles at *A* and *C* may evidently be eſteemed right angles, *AC* equal to *ac*, and therefore *Aa* equal to *Cc*. Hence in the right-angled triangle A*Aa*, we have A*a* : *Aa* :: R : coſ. A; and in the right-angled triangle C*Cc*, *Cc* : C*c* :: coſ. C : R; therefore A*a* × *Cc* : C*c* × *Aa* :: R × coſ. C : R × coſ. A, that is A*a* : C*c* :: coſ. C : coſ. A; or, $\dot{\overline{AB}} : \dot{\overline{BC}} :: \text{coſ.}\,C : \text{coſ.}\,A$, the third analogy. And by the ſame manner of reaſoning in the triangle DEF (fig. 15.) we have $\dot{\overline{DE}} : \dot{\overline{DF}} :: \text{coſ.}\,F : \text{coſ.}\,E$, therefore *per* lem. it will be $\dot{A} : \dot{C} :: \text{coſ.}\,BC : \text{coſ.}\,AB$, the fourth analogy. *Q. E. D.*

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* For it is demonſtrated, in moſt treatiſes on fluxions, that the fluxion of an arc is to that of its ſine as radius is to the coſine of this arc; and therefore when the radius is 1 we have $\frac{\dot{\overline{\text{ſine}}}}{\text{coſ.}}$ for the fluxion of the correſponding arc.

CASE. III. Fig. 17.

If any two of the ſides AB, BC, are conſtant, theſe analogies will obtain,

1. $\dot{B} : \dot{A} :: R \times \text{ſin.}\,AC : \text{ſin.}\,BC \times \text{coſ.}\,C$.
2. $\dot{B} : \dot{C} :: R \times \text{ſin.}\,AC : \text{ſin.}\,AB \times \text{coſ.}\,A$.
3. $\dot{B} : \dot{\overline{AC}} :: R^2 : \text{ſin.}\,C \times \text{ſin.}\,BC :: R^2 : \text{ſin.}\,A \times \text{ſin.}\,AB$.
4. $\dot{C} : \dot{\overline{AC}} :: \text{cot.}\,A : \text{ſin.}\,AC$.
5. $\dot{A} : \dot{\overline{AC}} :: \text{cot.}\,C : \text{ſin.}\,AC$.

That is,

1, 2. The fluxion of the angle formed by the two conſtant ſides is to the fluxion of either of the other two angles, as the rectangle under the ſine total and ſine of the variable ſide is to the rectangle under the ſine of the ſide oppoſite to the latter angle, and the coſine of the third angle adjacent to this ſide.

3. As the fluxion of the angle formed by the conſtant ſides is to the fluxion of its oppoſite ſide, ſo is the ſquare of the radius to the rectangle under the ſide of either of the other angles and the ſine of its adjacent ſide.

4, 5. As the fluxion of either of the angles adjacent to the variable ſide is to the fluxion of this ſide, ſo is the cotangent of the other adjacent angle to the ſine of the ſaid ſide.

DEMONSTRATION.

Let the angle ABC become AB*C*, and the ſides BC, B*C*, AC, A*C*, be produced to quadrants; then will E*e* evidently meaſure the variation of the angle at B, and G*g* that of the angle at A. From A deſcribe C*c*, and there will be formed the triangle C*c**C*, which on account of the ſmallneſs of the angle CA*C* may be eſteemed right-angled at *c*, as alſo the angle *c**C*C the complement of the angle C. From hence, and the ſimilar ſectors, we have theſe proportions,
E*e* : C*C* :: R. : ſin. BC; C*c* : G*g* :: ſin. AC : R; C*C* : C*c* :: R : coſ. C; the correſponding terms of which being multiplied, and the values of the arcs E*e*, G*g*, ſubſtituted in the product, we ſhall have
$\dot{B}$: $\dot{A}$ :: R × ſin. AC : ſin. BC × coſ. C, the firſt analogy. By a ſimilar proceſs we get, B : C :: R × ſin. AC : ſin. AB × coſ. A, the ſecond analogy. And by multiplying the correſponding

terms of theſe two proportions, $Ee : CC :: R :$ ſin. BC; $CC : cC :: R :$ ſin. C, we find $Ee : cC :: R^2 :$ ſin. BC × ſin. C, or, $\dot{B} : \dot{\overline{AC}} :: R^2 :$ ſin. BC × ſin. C; and if we ſuppoſe the angle B to flow towards A, we ſhall in like manner get $\dot{B} : \dot{\overline{AC}} :: R^2 :$ ſin. AB × ſin. A, which are the third analogies. By Caſe II. we have $\dot{B} : \dot{C} :: R \times$ ſin. AC : ſin. AB × coſ. A, and $\dot{\overline{AC}} : \dot{B} ::$ ſin. C × ſin. BC : R^2; therefore, by multiplying theſe two proportions together and reducing the product, we have, $\dot{\overline{AC}} : \dot{C} ::$ ſin. AC : cot. A; or by inverſion, $\dot{C} : \dot{\overline{AC}} ::$ cot. A : ſin. AC, the fourth analogy. And by the ſame method we find $\dot{A} : \dot{\overline{AC}} ::$ cot. C : ſin. AC, the fifth analogy. *Q. E. D.*

CASE IV. Fig. 16.

When any two of the angles B, C, are conſtant, the following analogies are derived,

1. $\dot{\overline{AC}} : \dot{\overline{AB}} ::$ ſin. B × R : ſin. C × coſ. BC.
2. $\dot{\overline{AC}} : \dot{\overline{BC}} ::$ ſin. B × R : ſin. A × coſ. AB.
3. $\dot{\overline{AC}} : \dot{B} ::$ coſec. AB : ſin. A :: coſec. BC : ſin. C.
4. $\dot{\overline{AB}} : \dot{B} ::$ cot. BC : ſin. B.
5. $\dot{\overline{BC}} : \dot{B} ::$ cot. AB : ſin. B.

That is,

1, 2. As the fluxion of the ſide oppoſite to the variable angle is to the fluxion of either of the other ſides, ſo is the rectangle under the radius and ſine of the variable angle to the rectangle under the ſine of the angle oppoſite to this other ſide and the coſine of the third ſide.

3. As the fluxion of the ſide oppoſite to the variable angle is to the fluxion of this angle, ſo is the coſecant of either of the other ſides to the ſine of the conſtant angle adjacent to this ſide.

4, 5. As the fluxion of either of the ſides, oppoſite to the conſtant angles is to the fluxion of the variable angle, ſo is the cotangent of the other ſide to the ſine of the ſaid angle.

DEMONSTRATION.

By applying this caſe to fig. 15. the two ſides DF, DE, will be variable, therefore by the laſt caſe we ſhall have, $\dot{D} : \dot{E} :: R \times$ ſin. FE : ſin. DE × coſ. F ; or, by ſubſtituting the correſponding values in the triangle ABC, $\dot{\overline{AC}} : \dot{\overline{AB}} :: R \times$ ſin. B : ſin. C × coſ. BC, the firſt analogy. In like manner may the ſecond analogy be obtained. And becauſe the ſquare of the radius divided by the ſine of an arc is equal to the coſecant of this arc, the third analogies will become $\dot{\overline{AC}} : \dot{B} :: R^2 :$ ſin.A × ſin.AB :: $R^2 :$ ſin.C × ſin.BC, and theſe proportions are eaſily inveſtigated by the triangle DEF, and the third analogy of Caſe III. The fourth and fifth analogies are immediately deduced by applying thoſe of Caſe III. to Fig. 15. *Q.E.D.*

I ſhall now ſhew the application of this theory by an example or two.

Example 1.

Required the hour of the day or night, by the obſerved altitude of a ſtar; and alſo the error of time, ſuppoſing the error in the obſerved altitude to be known.

Solution. In the ſpherical triangle PSZ, (fig. 18.) wherein P repreſents the pole, Z the zenith of the place of obſervation, and S the place of the ſtar, ZS will be the complement of the ſtar's altitude, PS the complement of its declination; PZ the complement of the latitude, and the angle ZPS a variable hour-angle contained betwixt the conſtant ſides PS, PZ. Now by Caſe III. we get, $\dot{P} : \dot{\overline{ZS}} :: R^2 : \text{ſin. } Z \times \text{ſin. } PZ$; but $\dot{P}$ is evidently the meaſure of the error in time, therefore we have $\dot{P} = \frac{R^2 \times \dot{\overline{ZS}}}{\text{ſin. } Z \times \text{ſin. } PZ}$: *Q. E. D.*

Cor. 1. From this equation it appears that, ſince $\frac{R^2}{\text{ſin. } Z}$ is a conſtant quantity, if the ſame error be ſuppoſed to be made in different obſervations in the ſame latitude, the error of time will not be altered, whatever the altitude of the ſtar be,

2. If the latitude be ſtill ſuppoſed the ſame, the error of time will be the leaſt when the ſtar is obſerved on the prime vertical; for then the angle at Z is a right angle, and therefore $\frac{R^2}{\text{ſin } Z}$ becomes barely R. And the error will be the leaſt poſſible if the obſervation be made at the equator when the ſtar is on the prime vertical, for then the expreſſion $\frac{R^2 \times \dot{\overline{ZS}}}{\text{ſin.} Z \times \text{ſin.} PZ}$ will evidently be a minimum, ſince ſin. Z × ſin. PZ will be the greateſt poſſible,

Example II.

Required the correction neceſſary to be made in determining the inſtant the ſun comes to the meridian from two equal altitudes; ſuppoſing the ſun's declination to undergo a little change during the interval of the obſervations.

Solution. If the ſun's declination was the ſame at both obſervations, half the interval would aſcertain the inſtant of noon; but as this can happen only at the ſolſtices, at all other times of the year this half interval muſt want a correction, by ſomething being ſubtracted when the ſun is in the aſcending ſigns, and added when in the deſcending ſigns; becauſe in the former caſe the ſun muſt evidently arrive later at the ſame altitude, and in the latter caſe ſooner. In order then to determine this correction, let P*s*, PS (fig. 19.) be two-hour circles very near to each other, and terminating in the ſame almacanter ESL, and the angle PS*s* will evidently expreſs how much later or ſooner the ſun comes to the ſame almacanter in the afternoon than in the morning; and therefore the half of it will expreſs the time which muſt be ſubtracted from, or added to, the half interval of the obſervations. Now as the change in declination for ſo ſmall an interval as three or four hours, may be ſafely taken for the naſcent variation or fluxion of the arc PS, and that the ſides PZ, SZ, of the triangle PSZ are conſtant, and the ſide PS variable; we ſhall have by Caſe III, as $\dot{P} : \dot{\overline{PS}} :: \text{cot. S} : \text{ſin. PS}$; or, (*per* ſpherics)

$$\dot{P} : \dot{\overline{PS}} :: \frac{\text{cot. PZ}}{\text{ſin. P}} \pm \frac{\text{cot. PS}}{\text{tang. P}} : \text{R}; \text{ hence}$$

$$\dot{P} = \frac{\dot{\overline{PS}}}{\text{R}} \times \overline{\frac{\text{tang. lat.}}{\text{ſin. P}} \pm \frac{\text{tang. dec.}}{\text{tang. P}}}.$$

The half of this expreſſion, reduced into time, is the required correction; the affirmative ſign taking place when the declination is ſoutherly, and the negative ſign when it is northerly. *Q. E. D.*

Remark. For the former of theſe examples (and indeed in a great meaſure for the theory of Trigonometrical Fluxions) I am indebted to Mr. Cotes's Treatiſe, *De æſtimatione errorum in mixtâ matheſi*; and the latter may be ſeen in De la Caille's Aſtronomy. A variety of examples of the applications of

Fluxional Analogies are alſo given in *Les Memoires de l'Academie*, 1774; and in M. De la Lande's Aſtronomy. But it may not be improper, however, to obſerve by way of caution to the young reader, that ſolutions obtained by this method cannot be depended on as ſtrictly true, unleſs the arcs under conſideration be in their evaneſcent ſtate; and therefore the nearer they approach thereto, the nearer will the ſolution approach to preciſion.

The following problems (among many others) are inveſtigated by the *Direct Method of Fluxions*. 1. The Maxima and Minima. 2. The finding of Tangents. 3. Determining the points of contrary Flexure in a Curve. 4. Finding the Evolutes of Curves. 5. Inveſtigating the Catacauſtic and Diacauſtic Curves, &c. &c. And the *Inverſe Method* is applied, 1. To the finding of the Lengths of Curve Lines. 2. To the Quadratures, or finding the Areas of Figures. 3. To inveſtigating the Surfaces of Solids. 4. To determining the Solidities of Bodies; in which is included all Menſuration. 5. To finding the Centers of Gravity, Percuſſion, and Oſcillation. 6. To inveſtigating the Law of Centripetal Force in a given Curve. And laſtly, to the ſolving of all Phyſical Problems whatſoever. Since then fluxions are univerſally applicable to all kinds of problems *, I would adviſe the young algebraiſt

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* There may be caſes propoſed which are too ſimple for the method of fluxions, ſuch as finding the areas of parallelograms, the ſuperficies and ſolidities of parallelopipedons, priſms, cylinders, &c. ſince a figure or ſolid ſimilar to theſe is evidently always aſſumed in the very nature of the fluxion; and therefore nothing can be inferred from thence. And, on the other hand, there may be caſes propoſed which ſeem to be inacceſſible by a fluxionary calculus. Such is the problem reſpecting the equality of the areas of the hyperbola and triangle of equal baſes, when the altitudes of both are ſuppoſed to be infinite. Here the method of increments anſwers ſuch

Fig. 14

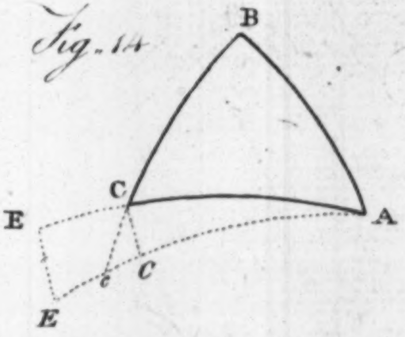


Fig. 15

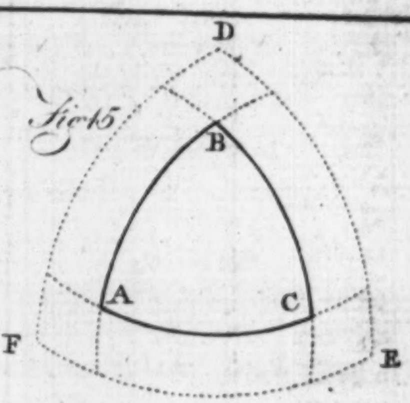


Fig. 16

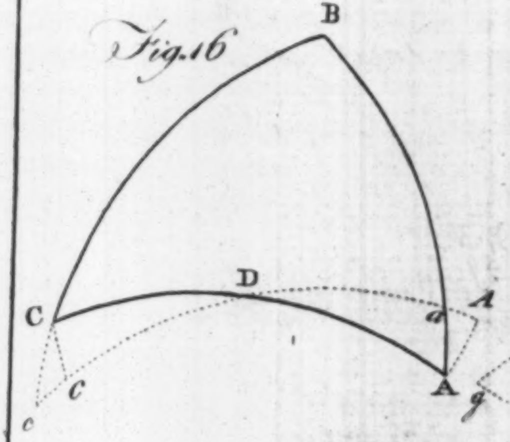


Fig. 17

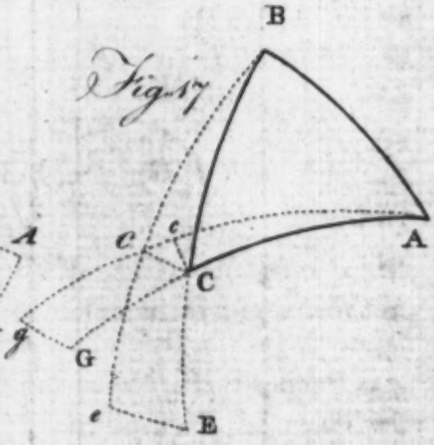


Fig. 19

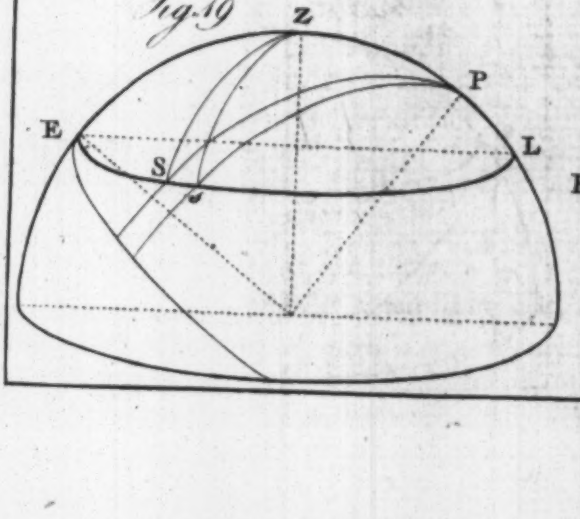
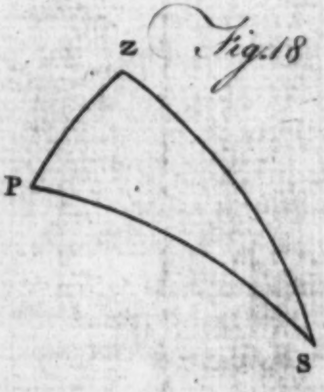


Fig. 18



algebraiſt to endeavour by all means to make himſelf thoroughly acquainted with them. I know of no book on the ſubject ſo well calculated to anſwer this end as Mr. Emerſon's. His forms for fluents are excellent; and nothing can more facilitate the learner's progreſs than his endeavouring to inveſtigate them.

The following problem having ſome time ago engaged my attention, by endeavouring to adapt an analytical ſolution thereto, inſtead of the trigonometrical one which has hitherto been made uſe of, I intended here to preſent the reader with the reſult of thoſe ſpeculations, not only becauſe the method of ſolution is new, but as the problem is applicable to ſome curious phœnomena in Aſtronomy, particularly in determining with great accuracy the periodic times of the diſappearance of Saturn's ring to an obſerver ſituated either on the earth or any of the planets; but as there is juſt publiſhed at Paris a treatiſe, entitled, *Eſſai ſur les Phénomènes relatifs aux diſparitions périodiques de l'Anneau de Saturne*, where this problem is ſolved in a more

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ſuch problems with great facility; when fluxions ſeem to be of no uſe. Unleſs, indeed, we may be allowed to reaſon after the manner of a well-known mathematical writer, in a caſe almoſt ſimilar, as follows: Since, by Hypotheſis, the laſt ratio of the areas, or fluents, is that of equality, their fluxions may be alſo eſteemed in that ratio; therefore, if by equating the fluxions, the value of the abciſs, or altitude, comes out *infinite*, we have then derived the ſame concluſion. Thus, in the caſe propoſed, let $t =$ the ſemitranſverſe, $c =$ the ſemiconjugate, and $x =$ the altitude of the triangle; then by the nature of the curve $\frac{c\dot{x}}{t}\sqrt{x^2-t^2}$ is the fluxion of the area of the hyperbola; and $\frac{cx^2\dot{x}}{2t\sqrt{x^2-t^2}} + \sqrt{x^2-t^2} \times \frac{c\dot{x}}{2t}$ is the fluxion of the area of the triangle. Theſe therefore being equated and reduced, become $x^2 = x^2 - t^2$; hence t is $= 0$; but ſince t is a conſtant quantity, it can therefore be only *comparatively* nothing to x. Conſequently x is infinite; for when a given finite quantity is 0 when compared with another quantity, the value of that other quantity muſt be infinite.

general manner, I have thought proper to reſerve my own ſolution for a future publication, and have here given an extract from the ſecond and third ſections of the above-mentioned work.

A Problem.

1. Suppoſe a body T (fig. 20.) revolves in a circle with an uniform motion; and in the ſame time another body R is uniformly carried in the right line $Rt'St$, paſſing through S the center of the circle; and that there is given the ratio of the velocity of the body T in the circle to that of the body R in the right line, and alſo the ſituation of the latter body in the right line at the inſtant the former is at t'. Required an expreſſion for the arch deſcribed by the body T from its commencement of motion at t', to the inſtant they are found in the ſame perpendicular to the right line $Rt'St$; alſo an expreſſion for the arch deſcribed by T when the body R arrives at S.

Solution.

2. Put $a = t'R$, $\frac{m}{n} =$ the ratio of the velocity of the body T to that of R, $r =$ the radius of the circle, $u = RQ$, and $z =$ the arch $t'TtP$. By the queſt. we have $n : m :: u : z$, $\therefore \frac{mu}{n} = z$; and ſince PQ is perpendicular to the right line tSt', PQ is the ſine, and SQ the coſine of the arch z; hence Qt' is equal to $r - \text{coſ}.z$. And by the conſtruction Qt' is equal $RQ - t'R = u - a$, therefore $u - a = r - \text{coſ}.z$; which,

becauſe $u = \frac{nz}{m}$, becomes $\frac{n}{m}z + \text{coſ.}\,z - r - a = 0$. This equation contains every caſe when the bodies T and R are in the ſame perpendicular to the line tSt'. When the body R is at S, u is equal $r + a$; therefore in this caſe the equation becomes $\frac{nz}{m} - r - a = 0$.

3. In the above equations it muſt be remembered that the ſign of $\text{coſ.}\,z$ is affirmative when the arch is between 0° and 90°, 270° and 360°; and negative when it is between 90° and 270°. That the arch z may be greater than 360°, and that the ſign of the coſine is the ſame for an arch of $360° + z$ as for the arch z.

4. Before we proceed to inveſtigate the unknown quantity z in theſe equations, we ſhall obſerve, that as the ſolution will be beſt exhibited by particular caſes, we ſhall therefore fix on ſuch conſtant quantities in the equations, as would ariſe from that part of aſtronomy to which the queſtion is particularly applicable, *viz.* the Phænomena of Saturn's Ring.

As the diſtances of the Earth and Saturn from the Sun are reſpectively as 1 : 9·5394, the length of the Earth's ſemidiameter will be the ſine of $6° \, 1' \, 2''$ in Saturn's orbit. Saturn then deſcribeth an arch of $6° \, 1' \, 2''$ from the inſtant the plane of his ring produced touches the Earth's orbit, to the inſtant it paſſes through the ſun; and as Saturn's periodic time in his orbit is 10761·615 days, he will be 179·83 days in running $6° \, 1' \, 2''$. The plane of his ring then taketh nearly 360 days to run the diameter of the earth's orbit; but as this is only the $\frac{113}{355}$th part of it, it is therefore $\frac{355}{113} \times 360$ days in running the whole orbit; but the earth takes 365·25 days to run the ſame diſtance, hence the velocity of the earth in its orbit is to the ſimultaneous velocity of the ring of Saturn (conſidered as in the diameter of the earth's orbit) as $\frac{355}{113} \times 360$ is to $365{\cdot}25$, $\because m : n :: \frac{355}{113} \times 360 : 365{\cdot}25 :: 3{\cdot}098 : 1$.

Hence the foregoing equations (the former of which, for the ſake of diſtinction, we ſhall hereafter call the *original equation)* in relation to Saturn will become

$$\frac{1}{3{\cdot}098} z + \text{coſ.}\, z - r - a = 0;$$

$$\frac{1}{3{\cdot}098} z - r - a = 0.$$

The firſt of theſe equations may have different values; the ſecond can only have one value.

5. Now to determine the number of real roots which equations of a ſimilar form with the original one may contain in general, we may eaſily perceive that they muſt depend on the value of a, and the ratio of m to n; and moreover that a cannot be greater than $\frac{n}{m}$ arch 360°, or $\frac{n}{m} \times \frac{2 \times 355}{113} \times r$, this being the ſpace the body R deſcribes in the right line during one revolution of T. For if a exceeded this quantity, it is evident, that it would become leſs after one revolution of T, which would therefore no ways affect the queſtion. It is alſo evident that the equation hath at leaſt one real value, ſince it is impoſſible for the body R to move in the right line Rt, while the body T revolves in the circumference of the circle t'TtP, without both being at ſome one inſtant in the ſame perpendicular. But to determine the number of the real values of z, we muſt inveſtigate ſuch values of a whereby the

original equation may contain two values of z, one real, the other imaginary, for it is by this double value that the roots of the equation in paſſing from imaginary to real, or from real to imaginary, may be determined.[*]

6. By

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[*] The demonſtration to this propoſition I ſhall give in M. Du Sejour's own words, " *Pour démontrer cette propoſition, je me ſers du théoréme connu de tous les Géometrés, par lequel on enſeigne à trouver la valeur analytique d'une fonction, lorſque la variable que cette fonction renferme, croît ou décroît d'une petite quantité.——Soit* φx *une fonction de la variable* x; $dx\,\Delta x$ *la différence de cette fonction;* $dx\,\Gamma x$ *la différence de* Δx; $dx\,\Psi x$ *la différence de* Γx, &c. *Soit enſuite* $\varphi\overline{x+z}$ *une fonction de* $\overline{x+z}$ *pareille à la fonction* φx, z *étant une trèſpetite quantité. On démontre que l'on a*

$$\varphi\overline{x+z} = \varphi x + \frac{z\Delta x}{1} + \frac{z^2\Gamma x}{1\times 2} + \frac{z^3\Psi x}{1\times 2\times 3} + \&c.$$

Appliquons ce principe au cas dont il s'agit. Nous avons en général $a = \frac{n}{m}z + \text{coſ.}z - r$. *Réduiſons en ſérie chacun des membres de cette équation au moyen du théoréme précédent. Il eſt aiſé de voir que ſi l'on ſuppoſe que, tandis que la variable* a *qui entre dans le premier membre de l'équation, croît d'une petite quantité* α, *la variable* z *qui entre dans le ſecond membre, croît d'une autre petite quantité* β, *l'on aura en vertu du théorême,*

$$a + \alpha = \frac{n}{m}z + \text{coſ.}z - r + \beta \times \frac{\frac{n}{m}r - \text{ſin.}z}{r} - \frac{\beta^2\text{coſ.}z}{1.2r^2} + \frac{\beta^3\text{ſin.}z}{1.2.3r^3}, \&c.$$

Mais par la ſuppoſition, $a = \frac{n}{m}z + \text{coſ.}z - r$; *de plus puiſqu'il s'agit des points particuliers qui répondent aux racines doubles de l'equation, l'on a* (§ 6.) $\frac{n}{m}r - \text{ſin.}z = 0$; *donc dans le cas où la quantité* β *eſt infiniment petite, l'on a*

$$2\alpha r^2 + \beta^2\,\text{coſ.}z = 0,$$

ou ce qui revient au même,

$$\beta = r\sqrt{\frac{-2\alpha}{\text{cof.}z}}.$$

Donc β *eſt demi-imaginaire & dépend du ſigne de* α. *Donc c'eſt par la valeur particulière de* a *correſpondante aux racines doubles de l'équation, que cette équation acquiert de nouvelles valeurs réeles, & réciproquement; puiſqu'en*

6. By the method *De Maximis et Minimis* we find that the equation hath a double value when $\frac{n}{m}dz + d \times \text{coſ.}\, z = 0$, or $\left(\text{becauſe } d \times \text{coſ.}\, z = -\frac{\text{ſin.}\, z.d.z}{r}\right) \frac{nr}{m} - \text{ſin.}\, z = 0$. Hence the equation will have double roots when the bodies T and R are in the ſame perpendicular, and the former hath deſcribed an arch the ſine of which is expreſſed by $\frac{nr}{m}$, which may be greater or leſs than 90°; or that arch augmented by 360 degrees, or any multiple thereof.

7. As the ſine of an arch can never exceed the radius, it is evident becauſe of the condition $\frac{nr}{m} - \text{ſin.}\, z = 0$, that the equation can only have double roots when n is leſs than m, or at moſt its equal. Let us ſuppoſe $n = m$, that is, let the bodies R and T have the ſame velocity, then we ſhall have $\text{ſin.}\, z = r$, and the correſponding values of z and $\text{coſ.}\, z$, will be *arch* 90°, and 0. Therefore in the ſuppoſition of $n = m$, Equations of a ſimilar form with the original one can only have double values when $a = \text{arch } 90^\circ - r$.

de-ça de cette valeur β eſt réel, & qu'au de-là de cette valeur β eſt imaginaire, ou réciproquement.

On démontre par la même analyſe, que l'équation ne peut acquérir de nouvelles valeurs réelles, que par les points particuliers qui répondent aux racines doubles; en effet s'il n'étoit pas queſtion de ces points, la quantité $\frac{nr}{m} - \text{ſin.}\, z$ *ne ſeroit pas nulle; & l'on auroit en vertu du théorême, dans la cas où β eſt une quantité infiniment petite,* $\alpha = \beta \times \frac{\frac{nr}{m} - \text{ſin.}\, z}{r}$. *D'où l'on voit que ſoit que l'on ſuppoſe α poſitif ou négatif, l'on aura toujours une valeur de β réelle."*

8. In order to exhibit the roots of the equation in the ſuppoſition of $n = m$, we ſhall repreſent the different values of a by the abciſſes of a curve, and the correſponding roots $\frac{nz}{m}$ by the ordinates of the ſame curve. The locus of the original equation will be a curve m, m', m'' (fig. 21.) having a point of inflexion at m', the ordinate to this point will touch the curve, and join three points thereof; hence the equation will have three equal roots, and the correſponding abciſs AP', or a, will be expreſſed by, arch 90° — r. If we take any other point m in the curve, the correſponding ordinate AP will be expreſſed by a ſimple equation, and therefore have only one real root. And in the ſuppoſition of n being greater than m, the curve will have no point of inflexion; for whatever the value of a is, the ordinate will be expreſſed by a ſimple equation, containing but one real root. If the curve be conſidered as purely geometrical, not only the part m, m', m'', but alſo the part m'', m''', m'''', equally reſolveth the queſtion; but though the inflexions are ſimilar, yet as the abciſſes AP''', AP'''', &c. are expreſſed by $360° + 90° - r$, $2 \times 360° + 90° - r$, &c. they are of no uſe in the aſtronomical part of the ſolution, as they exceed 360°, agreeable to what we obſerved in § 5. wherefore the part of the curve m, m', m'', can only ſatisfy the problem.

9. If n be leſs than m the locus of the equation will be a curve of various ſinuoſities (as in fig. 22, 23.) depending on the different relations of m to n. And *one* or *more* of theſe ſinuoſities (formed by the different returns of the curve) may be cut by the ſame ordinate line, according as the ratio of m to n approaches to, or recedes from, the ratio of equality.

10. To

10. To illuſtrate this, let us ſuppoſe the ratio of *m* to *n* as 10 to 9, and we ſhall have by the equation $\frac{nr}{m} - \text{ſin.}\, z = 0$, $z = \frac{9r}{10}$; from whence it appears that the angles $64^\circ, 9', 30''$, $115^\circ, 50', 30''$, anſwer the conditions of the queſtion; the coſine of the former arch being $+ \frac{r\sqrt{19}}{10} = + 43589$, and of the latter $- 43589$. Theſe values of the arch z and its coſine being ſubſtituted in the original equation, give

$$\frac{9}{10} \text{ arch } 115^\circ\, 50'\, 30'' - 43589 - 100000 - a = 0,$$

$$\frac{9}{10} \text{ arch } 64^\circ\, 9'\, 30'' + 43589 - 100000 - a = 0.$$

From the former of which equations we get $a = 38376$; and from the latter $a = 44369$. Other values of a may alſo be found from the equations

$$\frac{9}{10} \times \overline{360^\circ + 115^\circ, 50', 30''} - 43589 - 100000 - a = 0,$$

$$\frac{9}{10} \times 360^\circ + 64^\circ, 9', 30'' + 43589 - 100000 - a = 0,$$

from whence $a = 603863$, and 609856.

11. Since when $m : n :: 10 : 9$, the equation $\frac{n}{m} z + \text{coſ.}\, z - r - a = 0$, can only have three real roots when the value of a is between 38376 and 44369, it appears that if the ratio of the velocities of the two bodies T and R be as 10 to 9, and that if at the inſtant the body T is at t' the body R is alſo in the ſame point, they will meet but once in the ſame perpendicular; but if at that time R is at ſome diſtance from t' between 38376 and 44369, they will be three times in the ſame perpendicular; and if the diſtance be greater than the laſt number, they can meet but once.

12. If m have a much greater ratio to n than in the preceding ſuppoſition, the two angles z by which the original equation acquireth double roots, will be much farther removed from 90°, the one approaching nearer to 180°, and the other to 0°. If we ſuppoſe the angle z determined by the equation $z = \frac{nr}{m}$ to be always the leaſt of the angles reſulting from this relation, the original equation will in this caſe be of equal value with the following ones,

$$\frac{n}{m} \times \overline{180^\circ - z} - \text{coſ.}\, z - r - a = 0,$$

$$\frac{n}{m} z + \text{coſ.}\, z - r - a = 0,$$

The firſt of theſe equations ſhews the relation between the abciſs and ordinate at the extremity m of the ſinuoſity A $m'm$ (fig. 24.), and the ſecond expreſſeth *that* between the abciſs and ordinate at the other extremity of the ſame ſinuoſity m'. If then we ſuppoſe that the point of the curve m anſwereth to the abciſs $= 0$, the firſt equation will become

$$\frac{n}{m} \times \overline{180^\circ - z} - \text{coſ.}\, z - r = 0,$$

which by ſubſtituting for $\frac{n}{m}$ its equal $\frac{\text{ſin.}\, z}{r}$ becomes

$$\overline{108^\circ - z} \times \overline{\text{ſin.}\, z - r} \times \overline{r + \text{coſ.}\, z} = 0;$$

therefore the problem is determined, depending on the quadrature of the circle. From theſe equations we eaſily find

$$z = \text{arch } 46^\circ, 26', 0''; \ 180^\circ - z = 133^\circ, 34', 0'';$$

$$\frac{n}{m} = \frac{7246}{10000} = \frac{1}{1{\cdot}38}.$$

13. Hence it appears that if the ratio of the motions of the two bodies be as 10000 is to 7246, and that at the commencement of motion they are both at the point t', they will again

meet when T has deſcribed an arch of $133^{\circ}, 34', 0''$; which ſecond meeting will evidently be double. If the value of *a* is between 0 and 27640, there will be three meetings of the two bodies in the ſame perpendicular; but if *a* be greater than this number they can meet only once.

14. Not only when the value of *a* is between 0 and 27640, but when it hath the particular value 455262 (this being the laſt value of *a* which ſhould be conſidered, § 5.), the bodies R and T will meet three times in the ſame perpendicular; which will be evident if we conſider that this is the diſtance the body R has run during one revolution of T in its orbit. Let A (fig. 25.) be the origin of the curve A $m' m m'' m'''$, to which the abciſs is = 0; the abciſſes AP, AP'', reſpectively equal 27640, 455262. Now it is evident that the ſinuoſities of the curve are bounded by the ordinates Am, $P'm'$, $P''m''$; and as the ſinuoſity A$m'm$ is limited by the ordinate Am correſponding to the abciſs = 0, ſo is the ſinuoſity $m'' m''' m''$ limited by the ordinate $P''m''$ correſponding to the abciſs AP''. In this caſe, while the part of the curve $m'' m''' m''$ beginneth to appertain to the queſtion, the analogous part A$m'm$ ceaſeth to reſolve it.

15. If the ratio of *m* to *n* be ſtill farther diſtant from that of equality, the ſinuoſity A$m'm$ (fig. 26,) will not be limited by the ordinate Am correſponding to the abciſs = 0, as in the preceding caſe, but extendeth itſelf over the negative ſide of the abciſs; which nevertheleſs is one part of the curve which anſwers the conditions of the queſtion. In like manner

the ſinuoſity $m''' m'' m'''$ extends itſelf beyond the ordinate $P''' m'''$ anſwering to the laſt abciſs AP''', and is another part of the curve which ſatisfieth the queſtion. If in this caſe we denote the abciſs at the origin of the curve by 0; the laſt abciſs, (being that which reſults from § 5.) by a'''; the two values of a determined by the equations $\frac{n}{m} z + \text{coſ.}\, z - r - a = 0$, $\frac{n}{m} \times \overline{360° + 180° - z} - \text{coſ.}\, z - r - a = 0$, by a', a'', reſpectively; and the leaſt angle determined by the equation $z = \frac{nr}{m}$ by z; then will the two bodies meet three times in the ſame perpendicular when the value of a is between 0 and a, and between a'' and a'''; and only once when it is between a' and a''*.

16. We

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* This is a remarkable caſe, and ſhould be particularly attended to, becauſe it is only from the ſuppoſition of $a = 0$ that we can determine preciſely the number of real roots of the equation. For when we have diſcovered by the ratio of m to n that the equation is capable of containing a certain number of real roots, we only know in general that it hath eſſentially this number of real roots, except two; theſe two depending on the value of a. It is neceſſary then to have ſome diſtinguiſhing character, whereby we can aſcertain the reality of the two roots depending thereon. Now this characteriſtic is not conſtant, but changes in that particular caſe when the ſuppoſition of $a = 0$ giveth double roots to the equation. For inſtance, when the value of $\frac{m}{n}$ is between $\frac{1}{1}$ and $\frac{10000}{2173}$, the original equation may have either *three* real roots or only *one*, this circumſtance depending on the value of a. The particular value of $\frac{m}{n} = \frac{10000}{7246}$ giveth the equation two equal roots (§ 12.) in the caſe of $a = 0$; and it is in this point that the characteriſtic of a changes. When the value of $\frac{m}{n}$ is between $\frac{1}{1}$ and $\frac{10000}{7246}$, if we denote the two values of a determined by the

equation;

16. We ſhall now ſhew what muſt be the ratio of m to n, that the equation $\frac{n}{m}z + \text{coſ.}\,z - r - a = 0$ may have five real roots.

When the ſinuoſities of the curve are ſo far extended that the ſame ordinate touches the extremities of two of them as in fig. 27. the curve hath then eſſentially three real roots; and at the particular point correſponding to the abciſs $A\overset{\prime}{P}$ it hath five roots. At this point the curve paſſeth from *one* eſſential real root and two other poſſible ones to *three* eſſential real roots and two other poſſible ones; and to determine the particular condition which then taketh place, we muſt obſerve, that the equation $\frac{n}{m}z + \text{coſ.}\,z - r - a = 0$ expreſſeth the

equations $\frac{n}{m} \times \overline{180^\circ - z} - \text{coſ.}\,z - r - a = 0$, $\frac{n}{m}z + \text{coſ.}\,z - r - a = 0$, by $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, reſpectively, and by o, $\overset{\prime\prime}{a}$, and z, the ſame as in § 15, the bodies will meet *three* times when a is between $\overset{\prime}{a}$ and $\overset{\prime\prime}{a}$, and only *once* when a is between o and $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$ and $\overset{\prime\prime\prime}{a}$. But on the contrary when the value of $\frac{m}{n}$ is between $\frac{10000}{7246}$ and $\frac{10000}{2173}$, the characteriſtics of a are different; for in this caſe $\overset{\prime}{a}$ muſt be determined by the equation $\frac{n}{m}z + \text{coſ.}\,z - r - a = 0$; and $\overset{\prime\prime}{a}$ by the equation $\frac{n}{m} \times \overline{360^\circ + 180^\circ - z} - \text{coſ.}\,z - r - a = 0$; when the bodies will meet *three* times when a is between o and $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$ and $\overset{\prime\prime\prime}{a}$, and only *once* when a is between $\overset{\prime}{a}$ and $\overset{\prime\prime}{a}$.——It may perhaps appear to the reader, that when the characteriſtic of a changes, the particular values thereof might be determined from changing the order of theſe equations, that is by a reciprocal ſubſtitution of one caſe for the other; but from hence we ſhould find either that ſome values of a were negative, or ſome greater than thoſe which reſult from § 5. which therefore ought to be rejected. Hence we ſee the inconvenience attending ſuch a procedure.

relation between the abciſs and ordinate at the extremity m of the ſinuoſity $A\overset{\prime}{m}m$; and the equation $\frac{n}{m}\times\overline{360^\circ+180^\circ-z}-\text{coſ.}\,z-r-a=0$ determines the relation between the abciſs and ordinate at the extremity of the ſinuoſity $\overset{\prime\prime\prime}{m}\overset{\prime\prime}{m}\overset{\prime\prime\prime}{m}$. Now becauſe (by the hypotheſis) the two ſinuoſities have the ſame tangent $P\overset{\prime}{}\overset{\prime}{m}\overset{\prime\prime}{m}$, or rather becauſe they coincide at that point, we have at the ſame time the two following equations,

$$\frac{n}{m}z+\text{coſ.}\,z-r-a=0;$$

$$\frac{n}{m}\times\overline{360^\circ+180^\circ-z}-\text{coſ.}\,z-r-a=0;$$

which, by equating the two values of a derived from hence, give for the condition of the problem

$$\frac{n}{m}\times\overline{\frac{360^\circ+180^\circ}{2}-z}-\text{coſ.}\,z=0;$$

or, becauſe $\frac{n}{m}=\frac{\text{ſin.}\,z}{r}$,

$$\text{ſin.}\,z\times\overline{\frac{300\ +\ 180^\circ}{2}-z}-r\,\text{coſ.}\,z=0.$$

An equation depending on the quadrature of the circle. From hence we eaſily find

$x=$ arch $12^\circ, 32^\prime, 50^{\prime\prime}$, $\frac{n}{m}=\frac{2173}{10000}=\frac{1}{4{\cdot}603}$, and $a=2371$.

17. Hence it appears that if the ratio of the motions of the two bodies T and R be as 10000 is to 2173, and that when the body T is at $\overset{\prime}{t}$, R is diſtant 2371, they will meet five times in the ſame perpendicular. In all other circumſtances they can meet only three times.

18. We ſhall now enquire what the ratio of m to n ought to be, when a is ſuppoſed $= 0$, that the equation $\frac{n}{m} z + \text{coſ.}\, z - r - a = 0$ may have five real roots.

If the ratio of m to n be yet farther diſtant from that of equality, the ſinuoſity $\overset{'''}{m}\,\overset{''}{m}\,\overset{'''}{m}$ is not limited by the ordinate $\overset{'}{P}\overset{'}{m}\overset{''}{m}$, but paſſeth beyond it; and there is one particular caſe where it touches the ordinate correſponding to the abciſs $= 0$. To determine which we muſt obſerve that the equation

$$\frac{n}{m} \times \overline{360^\circ + 180^\circ - z} - \text{coſ.}\, z - r - a = 0,$$

expreſſeth the relation between the abciſs and ordinate at the extremity $\overset{''}{m}$ of the ſinuoſity $\overset{'''}{m}\,\overset{''}{m}\,\overset{'''}{m}$ (agreeable to § 16.) If then we ſuppoſe that the point $\overset{''}{m}$ in the curve hath its abciſs $= 0$, we ſhall have $a = 0$ in the laſt equation, therefore

$$\frac{n}{m} \times \overline{360^\circ + 180^\circ - z} - \text{coſ.}\, z - r = 0;$$

but $\frac{n}{m} = \frac{\text{ſin.}\, z}{r}$, hence by ſubſtitution,

$$\overline{360^\circ + 180^\circ - z} \times \text{ſin.}\, z - r \times \overline{r + \text{coſ.}\, z} = 0,$$

an equation depending ſtill on the quadrature of the circle. From hence we get

$$z = 12^\circ, 23', 48''; \quad 360^\circ + 180^\circ - z = 527^\circ, 36', 12'';$$

$$\frac{n}{m} = \frac{2147}{10000} = \frac{1}{4{\cdot}658}.$$

19. From whence it appears that if the ratio of the motions of T and R be as 10000 to 2147, and that they both begin to move at the ſame time from the point $\overset{'}{t}$, they will firſt be

in the ſame perpendicular at the commencement of motion, a ſecond and third time when T hath deſcribed two certain arches, and a fourth time when it hath deſcribed an arch of $527^{\circ}, 36', 12''$, from the point t', and this laſt meeting will be double. If the value of a be contained between 0 and 2341, determined by the equation $\frac{n}{m}z + \text{coſ.}\, z - r - a = 0$, there will be five meetings of the two bodies in the ſame perpendicular; but if a be greater than this laſt number they can meet only three times. But beſide the above limit for a, the particular value 134920 giveth yet five meetings of the bodies in the ſame perpendicular; this being the diſtance run by R during one revolution of T in its orbit, agreeable to what has been already obſerved in § 5. Let A (fig. 28.) be the origin of the curve, to which the correſpondent abciſs is $= 0$, and the abciſſes AP', AP'', reſpectively equal to 2314 and 134920. Here then it is evident that the ſinuoſities of the curve are bounded by the ordinates Am, $P'm'$, $P''m''$; and in the ſame manner as the ſinuoſity $m''\,m\,m''$ is limited by the ordinate Am of which the abciſs is 0, ſo is the ſinuoſity $m'''\,m''\,m'''$ limited by the ordinate $P''m''$ of which the abſciſs A is 134920. This is then the caſe where the part of the curve $m'''\,m''\,m'''$ beginneth to anſwer the queſtion conſidered aſtronomically, while the analogous part of the curve $m''\,m\,m''$ ceaſeth to reſolve it.

20. If the ratio of m to n continueth to be ſtill farther removed from the ratio of equality, the ſinuoſity of the curve $m''\,m\,m''$ will then not be limited by the ordinate Am correſponding to the abciſs $= 0$, but will extend itſelf over the negative ſide of the abciſs, as we have before obſerved in § 15,

and in a ſimilar manner will the ſinuoſity $m''' m'' m'''$ extend itſelf beyond the ordinate $P''m''$ correſponding to the laſt abciſs AP''. If in this caſe we denote the firſt abciſs by o; the laſt abciſs derived from § 5, by a'''; the two values of a reſpectively determined by the equations $\frac{n}{m} z + \text{coſ.}\, z - r - a = 0$, $\frac{n}{m} \times \overline{2 \times 360^{\circ} + 180^{\circ} - z} - \text{coſ.}\, z - r - a = 0$, by a', a''; and the leaſt angle deduced from the equation $z = \frac{nr}{m}$ by z; the two bodies will meet five times when a is between o and a', a'' and a'''; but will meet only three times when it is between a' and a''. If the value a be comprized between $\frac{10000}{2173}$ and $\frac{10000}{2147}$, and if a', a'', reſpectively repreſent the two values of a determined by the equations

$$\frac{n}{m} \times \overline{360^{\circ} + 180^{\circ} - z} - \text{coſ.}\, z - r - a = 0,$$

$\frac{n}{m} z + \text{coſ.}\, z - r - a = 0$, the bodies R and T will meet but three times when a is between o and a', a'' and a'''; and five times when between a' and a''.

21. If we would inveſtigate the ratio of m to n, ſo that the original equation might have ſeven real roots, we ſhould find for the condition of the problem

$$\text{ſin.}\, z \times \overline{\frac{2 \times 360 + 180^{\circ}}{2} - z} - r \,\text{coſ.}\, z = 0;$$

from whence we may eaſily find $z =$ arch $7^{\circ}, 22', 31''$;

$$\frac{n}{m} = \frac{1284}{10000} = \frac{1}{7{\cdot}791};\ a = 788.$$

And

Fig. 20

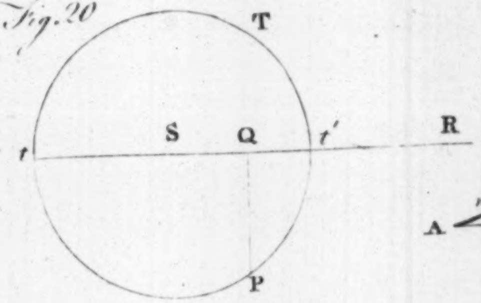


Fig. 21

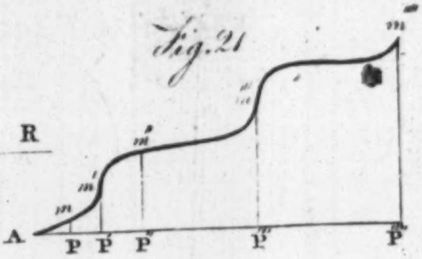


Fig. 22

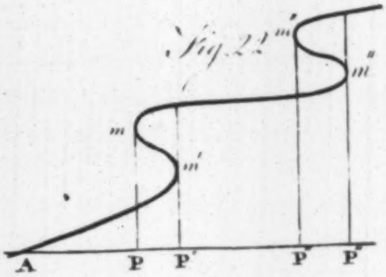


Fig. 23

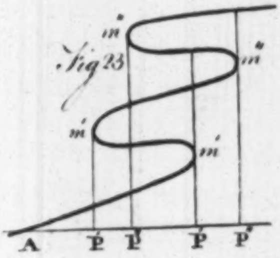


Fig. 24

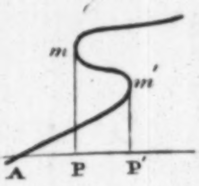


Fig. 25

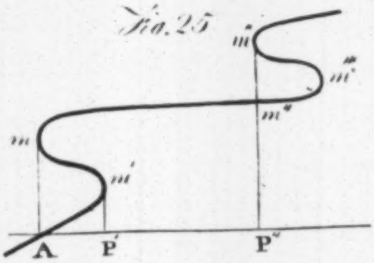


Fig. 26

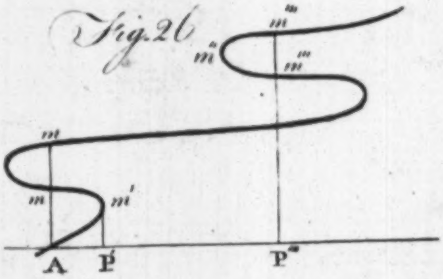


Fig. 27

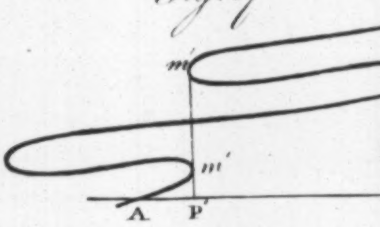
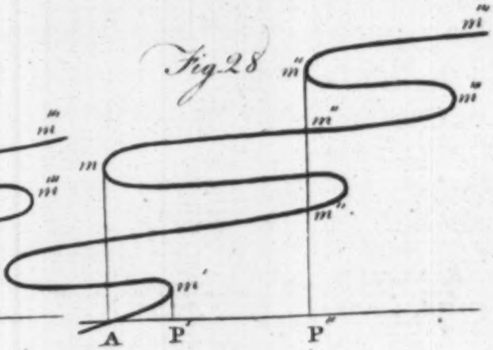


Fig. 28



And in the ſame manner we might proceed to determine the ratio of *m* to *n*, ſo that the original equation would be ſuſceptible of any aſſigned number of real roots.

22. We ſhall now exhibit in a general manner, how to determine in all caſes the number of real roots of the original equation.

Write down in the following order the equations which have been deduced in the preceding Articles,

$$1.\quad \frac{\text{ſin.}\,z}{r}\,z + \text{coſ.}\,z - r - a = 0;$$

$$2.\quad \frac{\text{ſin.}\,z}{r} \times \overline{180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

$$3.\quad \frac{\text{ſin.}\,z}{r} \times \overline{360^\circ + 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

$$4.\quad \frac{\text{ſin.}\,z}{r} \times \overline{2 \times 360^\circ + 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

$$5.\quad \frac{\text{ſin.}\,z}{r} \times \overline{3 \times 360^\circ + 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

$$6.\quad \frac{\text{ſin.}\,z}{r} \times \overline{4 \times 360^\circ + 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

&c.

Theſe ſhew the relation between the abciſs and ordinate correſponding to the extreme points of the different ſinuoſities of the curve. And if we ſuppoſe z to repreſent the leaſt angle determined by the equation $\frac{\text{ſin.}\,z}{r} = \frac{n}{m}$, we have ſeen that the compariſon of the firſt and ſecond equations ſheweth the circumſtance when the caſe propoſed hath *one* real root and beginneth to have *three*; and the equation to effect this is,

$$\text{ſin.}\,z \times \overline{\frac{180^\circ}{2} - z} - r\,\text{coſ.}\,z = 0.$$

The compariſon of the firſt and third equations giveth the circumſtance when the caſe propoſed hath eſſentially *three* real roots and beginneth to have *five*; the condition of which is,

$$\text{ſin.}\, z \times \overline{\frac{360 + 180^\circ}{2} - z} - r\, \text{coſ.}\, z = 0.$$

The compariſon of the firſt and fourth equations determineth likewiſe the circumſtance when the caſe propoſed hath eſſentially *five* real roots, and beginneth to have *ſeven*; and the condition of this is

$$\text{ſin.}\, z \times \overline{\frac{2 \times 360^\circ + 180^\circ}{2} - z} - r\, \text{coſ.}\, z = 0.$$

And as the analogy is eſtabliſhed, it is evident that if we put ν = the number of real roots the equation can have, we may eaſily determine the circumſtance when the equation propoſed beginneth to have this number of real roots, by the general expreſſion,

$$\text{ſin.}\, z \times \overline{\frac{\nu - 2}{2} \times 180^\circ - z} - r\, \text{coſ.}\, z = 0.$$

We may likewiſe determine the circumſtance when the equation hath *eſſentially* ν real roots, by the expreſſion

$$\text{ſin.}\, z \times \overline{\frac{\nu}{2} \times 180^\circ - z} - r\, \text{coſ.}\, z = 0.$$

In all theſe caſes the correſponding value of $\frac{n}{m}$ is given from the equations $\frac{n}{m} = \frac{\text{ſin.}\, z}{r}$.

23. From hence it would be no difficult matter to form a table, which ſhould ſhew by inſpection from the value of $\frac{n}{m}$, what number of real roots the equation contains. Now in order to aſcertain this matter truly, we muſt have regard to the

value of *a*, ſince two of the roots depend thereon. But it has been already ſhewn that when one value of *a* is denoted by o, the value of *a* determined by § 5. by $\overset{\prime\prime\prime}{a}$, and any two intermediate values of *a* by $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$; the combinations of the values *o*, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, $\overset{\prime\prime\prime}{a}$, will clearly exhibit the nature of *all* the roots of the equation. In the caſe, for example, where the equation hath eſſentially *one* real root, and may have *three* real ones, the values of $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, have been given by two of the three equations of § 22, that is by the equations (1) and (2) in one circumſtance, and by the equations (1) and (3) in another circumſtance; for the equation (1) always holds good, but the equations (2) and (3) cannot both hold good at the ſame time. We have ſeen alſo when the equation hath eſſentially *three* real roots, and may have *five* real ones, that the values of $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, have been given from two of the three equations (1), (3), (4), § 22, that is by the equations (3) and (1) in one circumſtance, and by the equations (1) and (4) in another circumſtance; the equation (1) always obtaining, while the equations (3) and (4) cannot both obtain at the ſame. As the analogy is then evident, we may conclude, that if ν repreſents the number of real roots the propoſed equation may contain, as before, the following equation will determine the values of $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$,

$$1.\ \frac{\text{ſin.}\,z}{r} + \text{coſ.}\,z - r - a = 0,$$

and the only uncertainty remaining is, which of the two following equations we muſt make uſe of,

$$2.\ \frac{\text{ſin.}\,z}{r} \times \overline{\overline{\nu - 2} \times 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

$$3.\ \frac{\text{ſin.}\,z}{r} \times \overline{\nu \times 180^\circ - z} - \text{coſ.}\,z - r - a = 0;$$

in which equations we muſt remember that z is determined by the equation $\frac{\text{ſin.}\,z}{r} = \frac{n}{m}$, and that $\frac{n}{m}$ is given in the queſtion.

24. This laſt difficulty may be eaſily obviated; for ſince the equations (2) and (3) § 23. cannot both obtain at the ſame time, if the equation (2) for inſtance, be that which we ought to uſe, the equation (3) will give eſſentially a value of a greater than $\overset{\prime\prime\prime}{a}$; but if the equation (3) be that we ſhould make uſe of, the equation (2) will give a value of a leſs than 0. We ſhould then immediately try the equation (2), and if it give a value of a greater than 0, this is the value we have denoted by $\overset{\prime}{a}$, and the equation (1) will give $\overset{\prime\prime}{a}$. If on the contrary the equation (2) giveth a value of a leſs than 0, it is of no uſe, and the equation (1) will determine $\overset{\prime}{a}$, as the equation (3) will give $\overset{\prime\prime}{a}$.

In the firſt caſe the propoſed equation will have $\nu - 2$ real roots when the value of a is between 0 and $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$ and $\overset{\prime\prime\prime}{a}$; and ν real roots when a is between $\overset{\prime}{a}$ and $\overset{\prime\prime}{a}$. In the ſecond caſe, the equation will have ν real roots, when a is between 0 and $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$ and $\overset{\prime\prime\prime}{a}$; and $\nu - 2$ real roots, when a is between $\overset{\prime}{a}$ and $\overset{\prime\prime}{a}$.

26. The table we have before mentioned for determining the number of real roots in the equation from the values of $\frac{n}{m}$, may be conveniently drawn up in the following form,

$\frac{n}{m} \sqsubset 1$

$\frac{n}{m} \sqsubset 1$

The equation hath eſſentially one real root, and can have no more, whatever the value of a is.

$\frac{n}{m} = 1$

One eſſential real root, and may have three, according to the different values of a.

$\frac{n}{m} = \frac{1}{4{\cdot}603}$

Three eſſential real roots, and may have five according to the different values of a.

$\frac{n}{m} = \frac{1}{7{\cdot}791}$

Five eſſential real roots, and may have ſeven, by the different values of a.

$\frac{n}{m} = \frac{1}{10{\cdot}951}$

And in the ſame manner may the table be continued as far as we pleaſe, by means of the equations of § 22.

The application of the ſolution of this problem to determining the times of the appearance and diſappearance of ſaturn's ring, will be obvious to any one who has attentively conſidered the ſubject; thoſe who have not, will not think their time ill ſpent in peruſing M. Du Sejour's curious treatiſe, from whence I have before obſerved this ſolution is chiefly extracted.——The requiſites for the application, as deduced from M. De le Lande's Aſtronomy, are as follow,

Mean diſtance of the earth from the ſun - - -	100000.
Mean diſtance of ſaturn from the ſun - - - -	953937.
Syderial revolution of the earth - - - -	365^d, 6^h, 9′.
Syderial revolution of ſaturn - - - -	10761, 14, 37.

Time

Time of the plane of the ring paſſing through the ſun, -	1774, *Jan.* 8th.
The earth's place in its orbit at that time, -	$3^{s}, 18^{\circ}, 23', 0''$.
Longitude of the aſcending node of ſaturn's orbit	3, 21, 43, 17.
Longitude of the node of the ring - - -	5, 17, 5, 0.
Inclination of the ring to the ecliptic - - -	31°, 20, 0.
Inclination of ſaturn's orbit to the ecliptic - -	2, 30, 20.
Longitude of thoſe points of the earth's orbit which the plane of the ring touches when produced,	2, 17, 5, 0. 8, 17, 5, 0.

Some new Geometrical Propoſitions, which will be often found uſeful in the ſolutions of Problems.

Propoſition I. Theorem. Fig. 29.

If from the middle of the chord of a given ſegment of a circle ADB, a perpendicular be drawn to D, and AD produced to E, ſo that DE be equal to AD, and through E another ſegment be deſcribed on AB; then, if a line be drawn from the end of the chord as AF, the part intercepted between the two peripheries GF will be always equal to GB. ——For DB is evidently = DE; and AFB is = AEB, alſo AGB = ADB (Euc. 21. 3.) therefore EDB = FGB, and the triangles BDE, BGF are ſimilar, conſequently BG = GF.

Propoſition II. Problem.

The ſame conſtruction being made as in the laſt propoſition, it is required ſo to draw the line AF, that either the ſum, difference, ratio, rectangle, ſum of the ſquares, or difference of the ſquares of AG and GF, may be of a given magnitude.

For the Sum.———With diſtance AF = the given ſum deſcribe an arch to cut the periphery of the greater ſegment in F, and join A, F; and AG + GF will be = the given ſum. This is evident; as alſo the limitation,——that the given ſum muſt not be greater than 2AD, nor leſs than AB.

The Difference.———With diſtance AL = the given difference of AG, GF, deſcribe an arch *a b*; and on AB deſcribe the ſegment of a circle containing an angle equal to the ſup. of DAB, (Euc. 33. 3.) and through the interſection L draw ALF; ſo will AG — GF = AL the given difference. ——For ſince AL is equal the given difference *per* con. GF will be equal GL; but GF = GB *per* con. therefore GL = GB; hence ALB = LGB + GBL (Euc. 32. 1) = ADB + DBA = ſup. of DAB.——The limitation is evident, ——the given difference muſt not be greater than the diameter of the leſſer circle.

The Ratio.———Divide AB in the given ratio of AG to GF in O, complete the circle ADB (Euc. 25. 3.), and produce DC to P; then through O draw FG, and through G draw AF; and AG will be to GF in the given ratio.——For ſince the arches AP, BP, are equal, the angle AGB is biſected by GP (Euc. 27. 3.) and therefore as AO : OB :: AG : GB :: AG : GF, *per* Euc. 3. 6. and conſtruction.

The ſame may be effected independent of the property of the circle.—Having divided AB in the given ratio in O, in any angle draw the indefinite line AH, and take AH to HI alſo in the given ratio; join IO, parallel to which draw HK, and from center K and diſtance KO deſcribe the arch OG; then draw AGF, and AF will be divided in the given ratio in G.——For by conſtruction AH : IH :: AO : OB, and becauſe of the parallels IO, HK, AH : HI :: KA : KO,

∴ KA

$\therefore$ KA : KO :: AO : OB, and by alternation and diviſion KA : KA − AO :: KO : KO − OB, that is KA : KO :: KO : KB, or KA : KG :: KG : KB; hence the triangles AKG, GKB are ſimilar (Euc. 6. 6.) and the other ſides will be proportional, or AG : GB :: AK : KG (KO) :: AH : HI, which are in the given ratio by conſtruction.

The Rectangle.———Let M repreſent the ſide of a ſquare equal to the given rectangle (Euc. 14. 2.). Find the centre of the leſſer circle V (Euc. 25. 3.) and on AB produced take AK equal to the diameter thereof; make the normal KT a third proportional to AK and M, and draw TG $\parallel$ to AK; then through G draw AGF, and $AG \times GF$ will be equal M^2. ——For through the center V draw GS, join SB, and demit the $\perp$ GR; then are the triangles AGR, SGB ſimilar (Euc. 21. 31. 3.) and AG : GR :: SG : GB, hence $AG \times GB = GR \times SG$; but $AK = SG$ *per* con. and $TK = GR$, $\therefore AG \times GB = AG \times GF = AK \times TK = M^2$ *per* conſtruction.——Limitation. M^2 muſt not exceed $AD \times DE$, or the ſquare of the ſemidiameter of the greater circle.

The Sum of the Squares.———Let $2M^2$ expreſs the ſum of the ſquares, and take $CQ^2 = M^2 - AC^2$; with diſtance CQ and center C deſcribe the arch QG, and through G draw AF, and $AG^2 + GF^2$ will equal $2M^2$.——For $AQ^2 = QC^2 + AC^2$ (Euc. 47. 1.) and $AQ^2 + QB^2 = 2AQ^2 = 2QC^2 + 2AC^2$; alſo $AG^2 + GB^2 = AC^2 + CG^2 + 2AC \times CR + AC^2 + CG^2 - 2AC \times CR$ (Euc. 12. 13. 2.) $= 2AC^2 + 2CG^2$; but $CG = CQ$ *per* con. therefore $AG^2 + GB^2 = AG^2 + GF^2 = 2AQ^2 = 2CQ^2 + 2AC^2 = 2M^2$ *per* conſtruction.——Limitation. M^2 muſt not be greater than $AC^2 + CD^2$, nor leſs than $\frac{1}{2}AB^2$.

The Difference of the Squares.——Let M^2 = the given difference of the ſquares, and take CR = to half a third proportional to AB and M; then draw RG ⊥ to AB and through the interſection G draw AF; ſo will $AG^2 - GF^2 = M^2$. ——For $AG^2 - GB^2$ is $= \overline{AR^2 + GR^2} - \overline{BR^2 + GR^2}$ (Euc. 47. 1.) $= AR^2 - BR^2 = \overline{AR + RB} \times \overline{AR - RB}$ (Euc. 5. 2) = (becauſe AR evidently exceeds RB by 2CR) $AB \times 2CR = M^2$ *per* conſtruction.——Limitation. M muſt not exceed AB.

Propoſition III. Theorem. Fig. 30.

If the baſe of a triangle ABC be produced both ways to D and E, ſo that CE = CB, and AD = AB; and the center of a circle F be found to paſs through the points D, B, E, (Euc. 5. 4.) and B,F be joined, the angle ABC will be biſected by BF.——For join BD, BE, FA, FD and FE, then is ACB = CBE + BEC = 2BEC (by con. and Euc. 32. 1.) = BFD (Euc. 20. 3.), and in like manner BAC = BFE; hence ABC = EDF + DEF = 2FDA; and becauſe the triangles BAF, DAF are evidently equal and ſimilar, ABF = FDA = $\frac{1}{2}$ABC, therefore ABC is biſected by BF.

Propoſition IV. Problem. Fig. 31.

There is given DC the diſtance of the indefinite line AB from the center of the given circle IFE; it is required ſo to draw a line through I, that the part intercepted between AB and the concave periphery of the circle may be of a given length.

On IE produced (if neceſſary) take IP equal to the given line, on which deſcribe the ſemicircle IOP. Make IS a mean proportional between CI and IE, and ⊥ to CE; draw SO ∥ to CE, and from I apply IL = SO, which produce to F, and

LF will be equal the given line.——For demit the normal OT, and join F, E. Then becauſe ICL, IFE are right angles, and CIL = FIE (Euc. 15. 1.) the triangles ICL, IFE are ſimilar; therefore CI : IF :: LI : IE, hence CI × IE = LI × IF; but SI^2 = CI × IE *per* con. alſo IP = the given line *per* con. ∴ IT × TP = TO^2 (Euc. 13. 6.) = SI^2 = LI × IF; and LI = IT (= SO) *per* con. conſequently IF = TP, and LF = IP the given line *per* conſtruction.

Propoſition V. Theorem. Fig. 32.

If two circles ACD, FDG touch each other, and another circle be any where deſcribed cutting both the former circles, as in the points A, C, G, F, the chords AC, FG being produced will meet the tangent drawn from the point of contact D in the ſame point B.——For in the circle ACGF, AB × BC = FB × BG; and in the circle ACD, AB × BC = BD^2; alſo in the circle FDG, FB × BG = BD^2 (Euc. 36. 3.) hence DB is a common tangent to both circles, and conſequently the lines AB, DB, FB meet in the ſame point B.

The converſe of this is alſo true.——That if two lines be any how drawn from a point B to cut a circle ACGF in C, A, G, F; and upon one of the chords as GF a circle be deſcribed FDG, and from the point B a tangent BD be drawn to this circle; then, if on ED produced the center of a circle be taken to paſs through the points A and C, it will alſo paſs through the point D, or touch the circle FDG in D.——For EDB is a right angle (Euc. 18. 3.) and therefore MDB is a right angle; and becauſe DM paſſes through the center of the circle M, and BD is drawn from a point B on a line AB cutting this circle, and moreover that AB × BC = FB × BG = BD^2, the line BD touches the circle M in D (Euc. 18. 38. 3.) conſequently the circle ACD touches the circle FDG in D.

Propoſition VI. Problem. Fig. 33.

It is required to find the center of a circle on a line AC given in poſition, which ſhall paſs through a given point on that line, and touch a circle given in magnitude and poſition.

From the given point A to the center of the given circle draw AB, make BF || to AC, and join A, F; then through E draw BC, and C will be the center of the required circle. ——For FEB = CEA, and becauſe of parallels EBF = ACE, or EFB = CAE, hence the triangles EBF, ACE are ſimilar; but BE = BF, therefore CA = CE.

Propoſition VII. Problem.

The ſame things being ſuppoſed as in the laſt propoſition, it is required to find the center of the circle when the given point A is not in the line AC.

The conſtruction is immediately deduced from propoſition V. For if HI (Fig. 32.) be the line given in poſition, C the given point, and FDG the circle given in magnitude and poſition, it is obvious, that if on the ⊥ CA, LA be made = CL, and a circle be deſcribed to paſs through A, C, and cut the given circle as in F, G, the lines AC, FG being produced, and a tangent drawn to the given circle from their interſection B, ED being produced till it meets HI, will give the center of a circle M which will paſs through C and touch the circle GDF.

Propoſition VIII. Problem.

To find the center of a circle to paſs through two given points, and cut off from a circle given in magnitude and poſition a given arch.

Let P, K be the two given points, (Fig. 32.) OGQ the given circle, and OQ the given arch. From N the center of the given circle deſcribe an arch to touch OQ; and through P, K deſcribe a circle cutting the given one in A, C; then produce AC, PK till they meet in B, from whence draw the tangent BF; and the points P, K, G, and F, will be in the periphery of the ſame circle, and which will cut off from the given circle the arch GF = the aſſigned one.——The demonſtration is the ſame as in Propoſition V.

Propoſition IX. Problem. Fig. 34.

To find the center of a circle on a line given in poſition in reſpect to a given circle, which may cut off from the given circle an aſſigned arch.

Let HI be the line given in poſition, and FGD the given circle. In any direction draw EP, and from T to F and G ſet off half the given arch; then will a circle deſcribed from P with diſtance PF or PG evidently cut off from the given circle an arch equal to that aſſigned.——This propoſition is unlimited; but is inſerted on account of its frequent uſe in geometrical conſtructions.

Propoſition X. Problem.

The ſame things being ſuppoſed as in the laſt propoſition, it is required to find the poſition of the center P when the circle is reſtricted to a given magnitude.

From the former given circle cut off DL = to the aſſigned arch; and perpendicular to the chord DL draw the indefinite line EK, to which from the point D or L apply DK = the ſemidiameter of the other given circle P; then from E to the line HI apply EP = EK, and P will evidently be the poſition of the center as required.——A demonſtration is needleſs.

From theſe few propoſitions may a great variety of Geometrical Problems be conſtructed, among which are the following ones in Burrow's Ladies' and Gentlemens' Diary for 1777*.

Queſt. 2.——The Data are reducible to this form,—— The baſe, the vert. angle, and the rectangle of the ſides.—— For ſuppoſe AEC the required triangle (Fig. 35.) ED $\perp$ to AC, and EB $=$ EC; there are given AB, and ECA $-$ EAC ($=$ AEB); alſo $m : n :: \text{EC (EB)} : \frac{n\text{EC}}{m}$, and $\text{AE} \times \frac{n\text{EB}}{m} =$ MN a given rectangle; therefore $\text{AE} \times \text{EB} = \frac{m}{n} \times \text{MN}$, a given quantity.——Hence this conſtruction,——On AB the given difference of the ſegments of the baſe, deſcribe a ſegment of a circle containing an angle equal to the given difference of the angles at the baſe, and by Propoſition III. find the point E ſo, that AE × EB may be equal to the given rectangle $\frac{m}{n} \times \text{MN}$; then on AB produced demit the perpend. ED, and make EC $=$ EB; ſo will AEC be the required triangle.——Which is too evident to need any farther demonſtration.

Limitation. $\frac{m}{n} \times \text{MN}$ muſt not exceed AB × BI.

Queſt. 4.——From the end of the given baſe AB (Fig .33.) draw BI equal thereto, making the angle ABI equal to the given one formed by the line drawn from the vertex to the baſe in D, and join A, I. From B with diſtance BE equal the given difference of the ſides deſcribe the circle EF; and by Prop. VI. draw BC ſo, that CE may be = CA; and ACB will be the required triangle.——For from the propoſition BE = CB — AC which is equal the given difference by con. and becauſe the angle ABI is equal that given in the queſt. by con. and CD || to BI (which is = BA) AD is evidently equal DC.

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* Theſe propoſitions were drawn up in 1775, which was before the commencement of Burrow's Diary; and before I had ſeen the Rev. Mr. Lawſon's *Diſſertation on the Geometrical Analyſis of the antients*, wherein I find we have both fallen upon the ſame method of conſtructing the 3d caſe of prop. II. being in effect the ſame with the 155 prop. of Pappus's 7th Book.——This I thought proper to mention, in order to prevent if poſſible, that too juſt charge againſt moſt modern writers——*plagiariſm*.

Remark. This queſtion ſeems to be improperly expreſſed, ſince from the other data the lines AD, DC admit of an equality; therefore they ſhould either have been reſtricted to *that*, or to a given ſum or difference.

Queſt. 10.——On AB the given intercepted line (Fig. 36.) deſcribe a circle ſuch, that the ſegment BAC may contain the given angle (Euc. 33. 3.); biſect AB in O, to which make OD perpendicular and equal the ſemidiameter of the given circle. Through D draw the indefinite line ML || to AB; and by Prop. IV. through F draw EC = to the given diſtance in the queſtion. Then complete the figure by drawing CAL, CBM; and from E with diſtance DO deſcribing the circle PH, *and the thing is done.*——For AB is equal the given intercepted line, and ACB the given angle *per* con. and ſince AB is biſected by the ⊥ OD, the arches AF, BF are equal (Euc. 30. 3.) and therefore ACB is biſected by CE (Euc. 27. 3.) which is equal the given diſtance *per* conſtruction; and becauſe DO is equal the ſemidiameter of the given circle, and DM || to AB, a circle deſcribed from center E and diſtance DO touches the line AB in P.——The min. limitation for AB in the data is obviouſly RS; but the max. is unlimited, for when AB becomes parallel to CL or CM it is infinite. Alſo from the conſtruction it appears that EC muſt not be greater than DQ; which are alſo both given.

After the ſame manner may one particular caſe of Queſt. 5. be conſtructed.———When BAC is a right angle; for then

NR is given, being equal the diameter of the latter circle. Or it may be conſtructed thus.——From any point Q on the indefinite line NR (Fig. 37.) with the given diſtance QA deſcribe the circle NAR and make QP ⊥ to NR; on NR deſcribe the circular ſegment NFR as in Prop. I. to which apply NF = 2CD — NR. Then on AQ deſcribe the circle QT, and through P draw AD = the given diſtance, and produce indefinitely AR, AN; laſtly, from center D and diſtance equal the ſemidiameter of the given circle deſcribe the arch BC, and it will touch the lines AB, AC, and NR. Therefore a tangent NR is drawn to the given circle D, cutting the other given circle in Q, and NQ is = QR. The reaſon of which is obvious from the conſtruction.———It is obſervable that this queſtion as it ſtands in the Diary may be more elegantly expreſſed——The perimeter. the line biſecting the baſe, and the vert. angle being given, to conſtruct the triangle;—which has never yet been done, if we may rely on the generality of Lawſon's Synopſis.——It may however be effected in a manner almoſt ſimilar with the laſt conſtruction, and ſtrictly geometrical.—For making BAC = the given vert. angle, AB = AC = half the ſum of the ſides, and drawing the indefinite line AD to biſect the ang. BAD, if we biſect AB in S, draw HI ⊥ to AD, and take SK a fourth proportional to $\overline{AS - HI}$, $\overline{AH \backsim AS}$, and AS, then will QN be always a fourth proportional to KS, AS, and $\overline{KS - AQ \backsim AS}$. Conſequently QN is given, and therefore if on NR = 2QN we deſcribe the ſegment of a circle containing the given vert. angle, and from Q apply QA = the given line biſecting the baſe, or find the point F as in prop. I. (NF being now known) and join AN, AR, then will ANR be the required triangle. The demonſtration whereof may be eaſily deduced from the preceding propoſitions; and the limitations in the data are evident from the above analogies.

As this book, from its profeſſed deſign, will naturally fall into the hands of thoſe who are not much read in the mathematics, I ſuppoſe it will not be unacceptable to ſuch to have a catalogue of thoſe books that are generally eſteemed the beſt on the ſubject, and which, it may reaſonably be expected, with aſſiduity and perſeverance, will ſoon make them proficients in the Mathematical Sciences.

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Fig. 29.

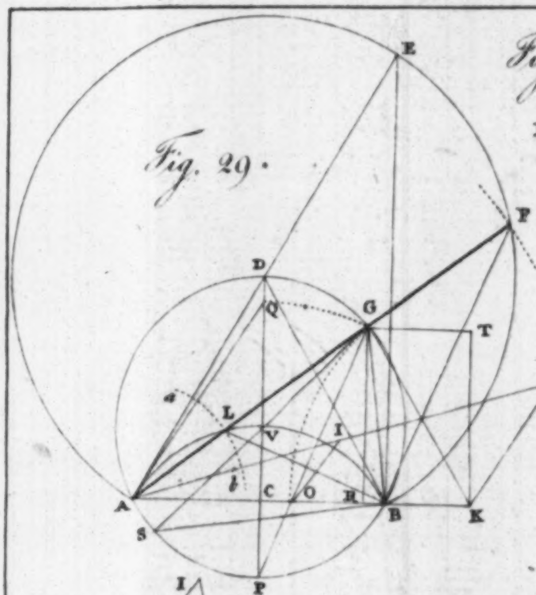


Fig. 30

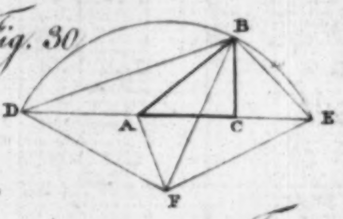


Fig. 31

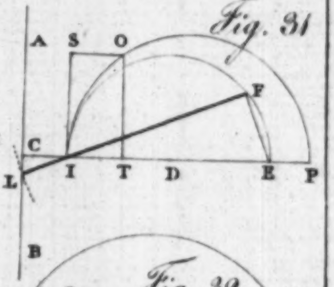


Fig. 32

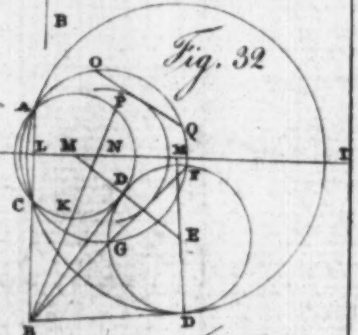


Fig. 33

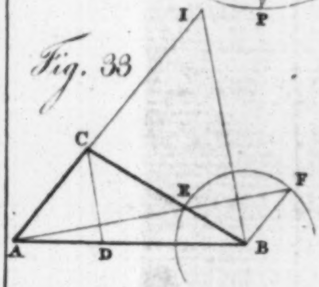


Fig. 35

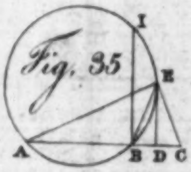


Fig. 34

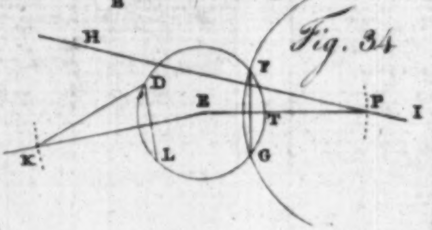


Fig. 37

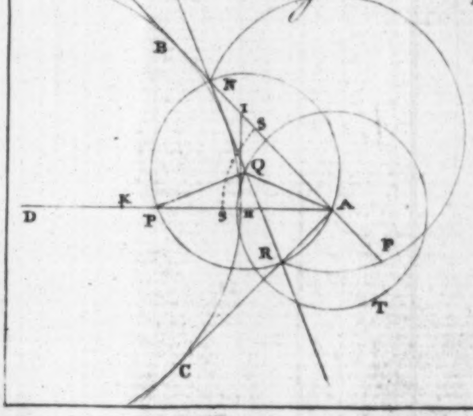
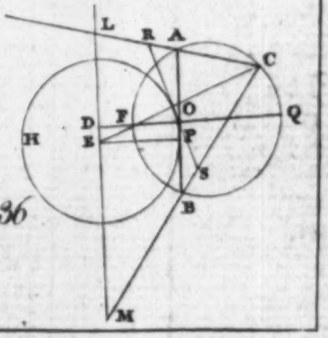


Fig. 36



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—Add the terminate part to the numerator of the given fraction; remove the decimal point, and daſh off the repetends as before; then increaſe the right-hand place by the difference of the *new* terminate part and that which was added, and the circulate will be correct.

Art. 32, ſhould have been thus expreſſed,—

—If the circulates to be added have the places of their repetends in a geometrical progreſſion, of any ratio, the Sum, &c.

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